Interior-Point Methods I: Barriers and the Central Path

Simplex uses basic solutions and pivots. The barrier approach studied here approaches an optimum through points with strictly positive standard-form variables.

Our LP: minimize $f = -2u - v$, subject to $u + v \leq 4,\ u \leq 3,\ u, v \geq 0.$

We will answer three questions: Why stay inside? How do we choose the next point? How do we know we are close enough?

A single two-variable LP will connect the pictures, calculations, and optimality conditions.

How methods for linear programming developed

Milestone	What changed?
1947 · Dantzig	Simplex solves LPs by changing bases. It is effective in practice, although some pivot rules need exponentially many pivots on certain LPs.
1979 · Khachiyan	The ellipsoid method establishes that LP can be solved in polynomial time. This guarantee does not make it the fastest practical method.
1984 · Karmarkar	A polynomial-time interior-point method opens a new direction for LP algorithms.
Late 1980s onward	Barrier and primal–dual interior-point methods turn the interior approach into widely used computational methods.

> **Polynomial time:** the worst-case running-time bound grows at most polynomially with the number of bits used to describe the LP.

Logarithmic barrier ideas predate 1984. Our next pages develop a logarithmic barrier method for solving an LP.

Where the LP methods fit

Method family	Main operation
Simplex: primal, dual, network	Change a basis. Network simplex uses the graph to perform the pivot calculations.
Ellipsoid	Repeatedly shrink a search region using inequalities. Its role here is the polynomial-time breakthrough.
Interior-point methods	Use positive variables while approaching optimality. Our barrier method computes directions by solving linear equations.
Primal–dual hybrid gradient (PDHG)	Update primal and dual variables using matrix–vector products and simple operations, such as replacing negative components by zero. Details come later.

> **Specialized network algorithms** exploit graph structure: for example, maximum-flow and Hungarian algorithms.

Column and constraint generation solve a large LP through smaller LPs. Those smaller problems still need an LP solution method.

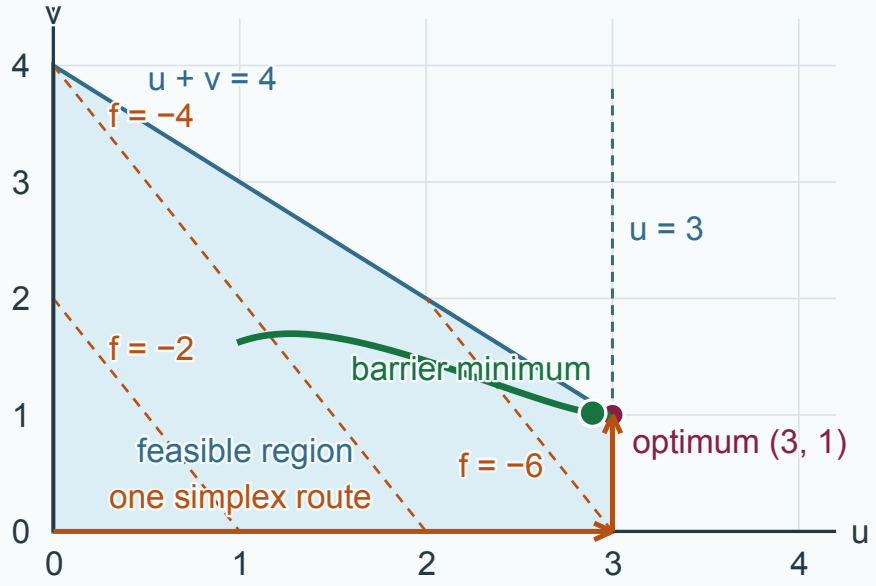
Must we follow edges to approach the optimum?

Our LP: minimize $f = -2u - v$, subject to $u + v \leq 4,\ u \leq 3,\ u, v \geq 0$.

Orange arrows: one simplex route is (0,0) → (3,0) → (3,1), along the boundary of the feasible polygon.

Green curve: each point minimizes an objective with a **barrier**: a penalty that grows near the boundary. The positive parameter μ controls that penalty. This family of minimizers is called the **central path**.

These green points are not successive algorithm steps. We will learn how to compute approximations to them.



μ = 0.1 · u = 2.893, v = 1.016
LP objective f = -6.802; slacks = (0.091, 0.107)

Green: barrier minima as μ decreases. Orange dashed lines: equal LP objective values. Solid orange arrows: one simplex route.

First locate the optimum we will approach

Our LP: minimize $f = -2u - v$, subject to $u + v \le 4,\ u \le 3,\ u, v \ge 0$.

$$-2u - v = -7 + (4 - u - v) + (3 - u).$$

Both parenthesized terms are nonnegative at every feasible point. Therefore every feasible objective value is at least –7.

$$(u^*, v^*) = (3, 1), \qquad f^* = -7.$$

Equality requires u + v = 4 and u = 3, so this optimum is unique. The two unused-capacity quantities are zero at the optimum.

What does “inside” mean for this problem?

Our LP: minimize $f = -2u - v$, subject to $u + v \le 4,\ u \le 3,\ u, v \ge 0$.

Variable order: $x = (u, v, s_1, s_2)^\top$, where $s_1 = 4 - u - v$ and $s_2 = 3 - u$.

$$u > 0, \quad v > 0, \quad s_1 = 4 - u - v > 0, \quad s_2 = 3 - u > 0.$$

A **strictly feasible point** here satisfies the equalities after introducing slacks and has all four components of x positive. For example, $(u, v) = (1, 1)$ gives $x = (1, 1, 2, 2)$.

Strict positivity refers to the variables in standard form. Equality constraints still hold exactly; “inside” does not mean ignoring them.

Make approaching zero increasingly expensive

For one positive quantity t, the function $-\log t$ tends to $+\infty$ as $t \downarrow 0$. A logarithmic **barrier** adds this term to discourage reaching the boundary.

$$\lim_{t \downarrow 0} (-\log t) = +\infty, \qquad t > 0.$$

> Apply a barrier to each quantity that must remain positive: u, v, 4 – u – v, and 3 – u.

The multiplier $\mu > 0$ controls the strength of the barrier. The logarithm need not be positive everywhere; its growth near zero is what keeps a finite minimizer away from the boundary.

Replace the LP by a smooth problem inside its region

Our LP: minimize $f = -2u - v$, subject to $u + v \le 4,\ u \le 3,\ u, v \ge 0$.

$$F_\mu(u, v) = -2u - v - \mu\big[\log u + \log v + \log(4 - u - v) + \log(3 - u)\big].$$

For a fixed $\mu > 0$, minimize F_μ over $u > 0, v > 0, u + v < 4, u < 3$.

The original linear objective favors larger u and v. The four logarithms oppose getting too close to any of the four boundaries.

This is an auxiliary nonlinear optimization problem. Its objective F_μ is different from the original LP objective $f = -2u - v$.

See the barrier on a one-dimensional slice

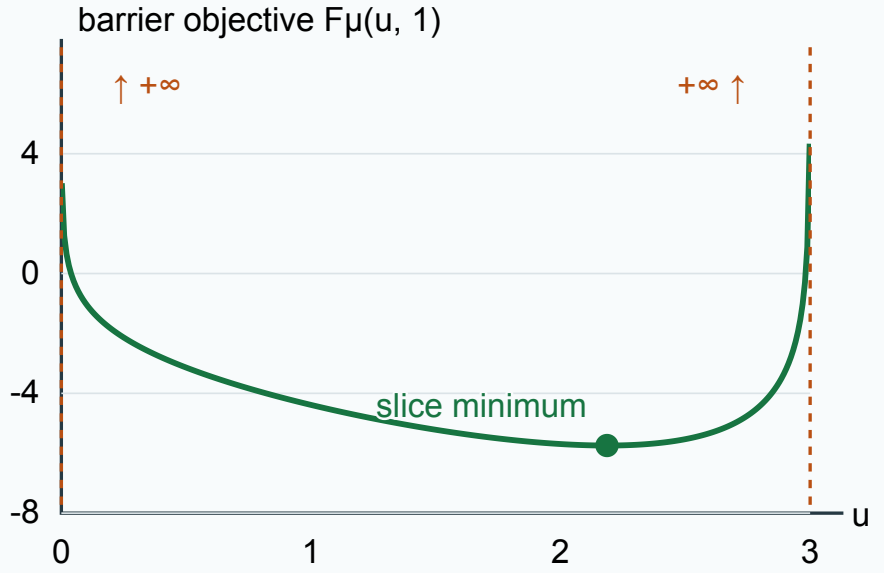
Fix $v = 1$. Feasibility then requires $0 < u < 3$.

$$F_\mu(u, 1) = -2u - 1 - \mu \log u - 2\mu \log(3 - u).$$

Both unused-capacity quantities equal 3 – u on this slice, which explains the coefficient 2.

> Move μ to see the minimum shift. The curve rises near both endpoints; a finite positive μ never places its minimizer on an endpoint.

This is only the slice v = 1, not the complete two-dimensional central path.



μ = 1 · Keep v = 1. The slice minimum has u = 2.186.

This is a one-dimensional slice with v fixed at 1. Its minimum need not be the minimum over all feasible (u, v).

A smaller barrier gives the original objective more influence

Our LP: minimize $f = -2u - v$, subject to $u + v \leq 4,\ u \leq 3,\ u, v \geq 0$.

Parameter	What is minimized?	Meaning
$\mu = 1$	$f - \sum \log x_j$	Boundary avoidance has substantial influence.
$\mu = 0.1$	$f - 0.1 \sum \log x_j$	The linear objective has more influence.
$\mu \downarrow 0$	A sequence of barrier problems	The minimizers approach an LP optimum.

We change μ between barrier problems. Within one barrier problem, μ is fixed while we compute its minimizer.

Reducing μ does not mean taking a larger geometric step. We still need an algorithm to find the new minimizer.

The central path is a family of barrier minimizers

Let $x(\mu)$ be the minimizer of the barrier problem with parameter $\mu > 0$. The collection of these points is called the **central path**.

Variable order: $x = (u, v, s_1, s_2)^\top$, where $s_1 = 4 - u - v$ and $s_2 = 3 - u$.

> For our bounded polygon, each positive μ has a unique barrier minimizer. The appendix proves existence and uniqueness.

The path is a reference for the algorithm. The numerical algorithm generally computes points near the path, not exactly on it.

Appendix: why the barrier has one minimizer.

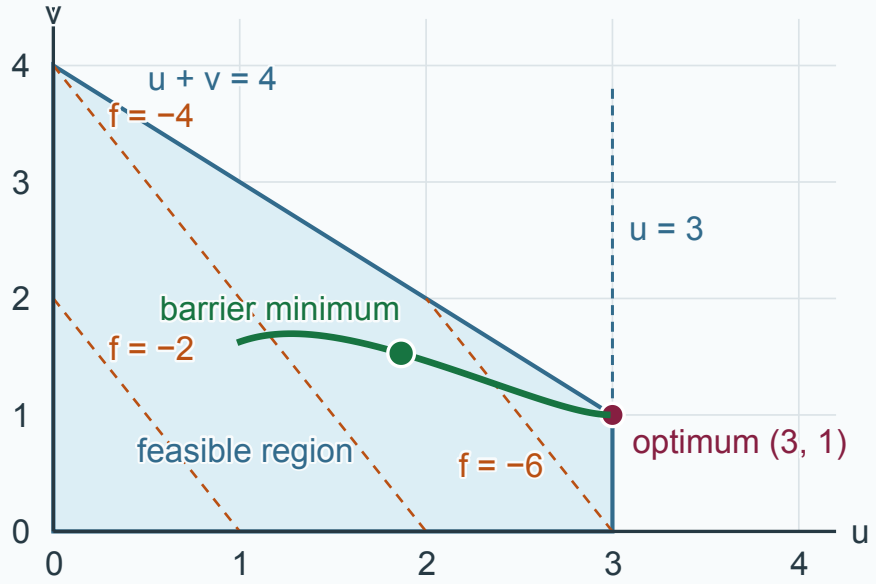
Move along the central path

Our LP: minimize $f = -2u - v$, subject to $u + v \leq 4,\ u \leq 3,\ u, v \geq 0$.

Lower μ. The point approaches (3,1), while all four standard-form variables remain positive.

The objective contours are lines of equal original LP cost. A smaller f means a better objective value.

These are numerically computed barrier minima for different μ values. They are not successive Newton steps.



μ = 1 · u = 1.866, v = 1.529
LP objective f = -5.262; slacks = (0.605, 1.134)

Green: barrier minima as μ decreases. Orange dashed lines: equal LP objective values.

Read the numerical path together with its slacks

Our LP: minimize $f = -2u - v$, subject to $u + v \le 4,\ u \le 3,\ u, v \ge 0$.

Variable order: $x = (u, v, s_1, s_2)^\top$, where $s_1 = 4 - u - v$ and $s_2 = 3 - u$.

μ	u	v	s_1 = 4 – u – v	s_2 = 3 – u
1	1.86614	1.52924	0.60462	1.13386
0.1	2.89318	1.01579	0.09104	0.10682
0.01	2.98993	1.00017	0.00990	0.01007

Both slacks approach zero. The variables u and v approach 3 and 1; they do not approach zero.

Each row is a different barrier problem solved to its minimum. Numbers are rounded, so sums may differ in the last digit.

Practice: what changes when μ decreases?

Our LP: minimize $f = -2u - v$, subject to $u + v \le 4,\ u \le 3,\ u, v \ge 0.$

Use the plot or numerical table. Work for about 3 minutes.

1. Does the barrier minimizer remain feasible for the original LP?
2. Must every component of x approach zero?
3. Is the central-path point already the LP optimum when μ = 0.01?

Justify each answer using the inequalities or the displayed slack values.

Positive slacks explain the difference from an exact optimum

1. Yes. The point satisfies both original inequalities and nonnegativity; in standard form it satisfies Ax = b with x > 0.

2. No. In this example u → 3 and v → 1, while s_1 and s_2 approach zero.

$$f - f^* = s_1 + s_2 > 0 \quad \text{for every positive } \mu.$$

3. At μ = 0.01, f is close to −7 but is still greater than −7. Interior-point methods normally stop at a prescribed numerical accuracy.

Write the same example in standard form

Variable order: $x = (u, v, s_1, s_2)^\top$, where $s_1 = 4 - u - v$ and $s_2 = 3 - u$.

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \\ s_1 \\ s_2 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 4 \\ 3 \end{pmatrix}}_{b}, \qquad c = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\min c^\top x \quad \text{s.t. } Ax = b, \quad x \geq 0.$$

There are n = 4 nonnegative variables, so the standard-form barrier has four logarithms. The equality rows are the two original capacity equations.

A short derivative reminder before Newton's method

A derivative measures how a function changes for a small change of its input. A second derivative measures how its slope changes.

$$\frac{d}{dt}(-\mu \log t) = -\frac{\mu}{t}, \qquad \frac{d^2}{dt^2}(-\mu \log t) = \frac{\mu}{t^2} > 0.$$

> At an interior minimum of a differentiable one-variable function, its derivative is zero.

$$\min_{t>0} \; t - \mu \log t : \quad 1 - \frac{\mu}{t} = 0 \;\Rightarrow\; t(\mu) = \mu.$$

The original LP min t, t ≥ 0 has optimum 0. Its barrier minimizer approaches 0 as μ decreases.

With two variables, collect slopes and curvatures

For a smooth function $F(u, v)$, the **gradient** collects its two partial derivatives. The **Hessian** collects the derivatives of those slopes.

$$g = \nabla F = \begin{pmatrix} F_u \\ F_v \end{pmatrix}, \qquad H = \nabla^2 F = \begin{pmatrix} F_{uu} & F_{uv} \\ F_{vu} & F_{vv} \end{pmatrix}.$$

For a small change d = (d_u,d_v), approximate the function by a quadratic:

$$F((u, v) + d) \approx F(u, v) + g^\top d + \tfrac{1}{2} d^\top H d.$$

> The gradient supplies the linear change; the Hessian supplies the curvature correction. We will compute both for the barrier.

Newton's direction minimizes the local quadratic

$$q(d) = F(u,v) + g^\top d + \tfrac{1}{2} d^\top H d.$$

A symmetric H is **positive definite** if $d^\top H d > 0$ for every nonzero d: the quadratic curves upward in every direction.

$$\nabla q(d) = g + Hd = 0 \quad \Longleftrightarrow \quad Hd = -g.$$

For positive-definite H, this is the unique quadratic minimum. Solve the linear equations for d; an explicit inverse is unnecessary.

$$(u,v)_{\text{new}} = (u,v) + \alpha d, \qquad 0 < \alpha \le 1.$$

The step length α may need to be smaller than 1 because the quadratic only approximates the barrier locally.

Compute the barrier gradient and Hessian

Our LP: minimize $f = -2u - v$, subject to $u + v \le 4,\ u \le 3,\ u, v \ge 0$.

Write $s_1 = 4 - u - v$, $s_2 = 3 - u$. Differentiate with μ fixed:

$$g = \begin{pmatrix} -2 - \mu/u + \mu/s_1 + \mu/s_2 \\ -1 - \mu/v + \mu/s_1 \end{pmatrix}.$$

$$H = \mu \begin{pmatrix} u^{-2} + s_1^{-2} + s_2^{-2} & s_1^{-2} \\ s_1^{-2} & v^{-2} + s_1^{-2} \end{pmatrix}.$$

The off-diagonal entries come from the shared slack 4 – u – v. Changing either variable changes that same slack.

One Newton calculation, with actual numbers

Our LP: minimize $f = -2u - v$, subject to $u + v \le 4,\ u \le 3,\ u, v \ge 0$.

Fix $\mu = 1$ and start at $(u, v) = (1, 1)$, so $s_1 = s_2 = 2$.

$$g = \begin{pmatrix} -2 \\ -3/2 \end{pmatrix}, \qquad H = \begin{pmatrix} 3/2 & 1/4 \\ 1/4 & 5/4 \end{pmatrix}.$$

$$Hd = -g \quad \Longleftrightarrow \quad \begin{cases} 6d_u + d_v = 8, \\ d_u + 5d_v = 6, \end{cases} \qquad d = \begin{pmatrix} 34/29 \\ 28/29 \end{pmatrix}.$$

The full step gives $(u, v) = (63/29, 57/29)$. Their sum is $120/29 > 4$: the full step is infeasible.

Shorten the Newton step before it reaches the boundary

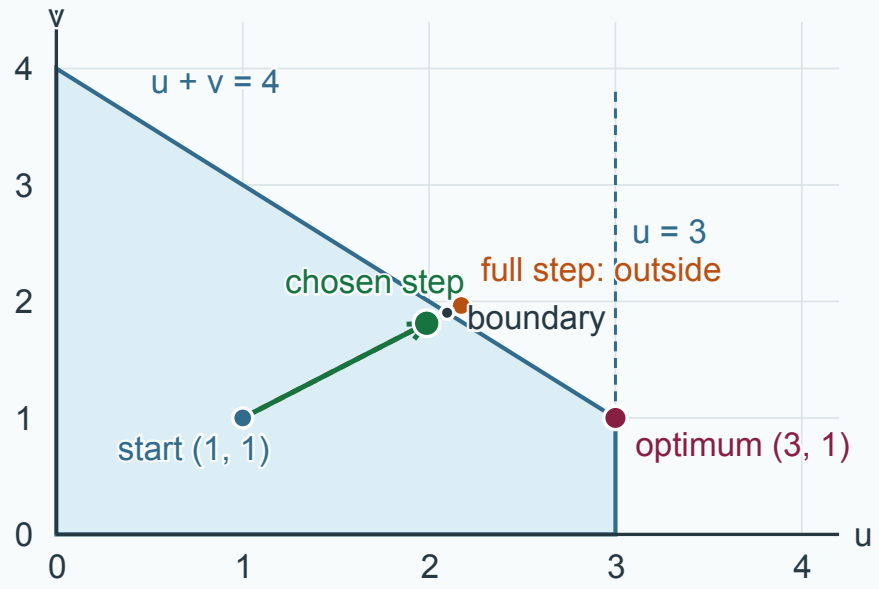
At $(1, 1)$, with $\mu = 1$, use $d = (34/29, 28/29)$.

$$s_1(\alpha) = 2 - \tfrac{62}{29}\alpha > 0 \quad \Rightarrow \quad \alpha < \tfrac{29}{31}.$$

The u = 3 boundary is farther away. The first boundary is u + v = 4.

> Take $\alpha = 0.9(29/31) = 261/310$. Then $(u, v) = (308/155, 281/155)$, and $s_1 = 1/5 > 0$.

This step also decreases the barrier. In general, check decrease and shorten further if needed.



α = 0.842 · (u, v) = (1.987, 1.813)
4 − u − v = 0.2; all four variables are positive.

The dashed extension shows the same Newton direction. Shortening the step preserves positivity.

Two nested tasks: solve, then reduce μ

For a fixed μ: start strictly feasible; compute g and H; solve Hd = –g; choose a step that preserves positivity and decreases the barrier; repeat until sufficiently close to the barrier minimum.

Then reduce μ: use the previous point as the starting point for the next barrier problem. Repeat until the original LP has a sufficiently small certified error.

The interactive central path shows the target minimizers. Newton steps are how we compute approximations to those targets.

Appendix: a precise decrease test and backtracking rule.

The stationary equations reveal dual variables

Our LP: minimize $f = -2u - v$, subject to $u + v \leq 4,\ u \leq 3,\ u, v \geq 0$.

At a barrier minimum, set both gradient components to zero. Define $p_1 = -\mu/s_1$ and $p_2 = -\mu/s_2$:

$$-2 - p_1 - p_2 = \mu/u > 0, \qquad -1 - p_1 = \mu/v > 0.$$

These equations imply $p_1 + p_2 < -2$ and $p_1 < -1$. Also $p_1 < 0$ and $p_2 < 0$. These are exactly strict versions of the four dual inequalities.

This calculation uses stationarity at the barrier minimum. At an arbitrary interior point, defining p this way need not satisfy the first two dual inequalities.

Keep the complete primal and dual visible

Primal: minimize

$$-2u - v$$

$$\begin{aligned} u + v + s_1 &= 4, \\ u + s_2 &= 3, \\ u, v, s_1, s_2 &\geq 0. \end{aligned}$$

Dual: maximize

$$4p_1 + 3p_2$$

$$\begin{aligned} p_1 + p_2 &\leq -2, \\ p_1 &\leq -1, \\ p_1 &\leq 0, \\ p_2 &\leq 0. \end{aligned}$$

The equality-row dual variables p_1, p_2 have no separate sign restriction. The last two inequalities come from columns s_1, s_2.

Define **dual slacks** $r = c - A^\top p$. The symbol r distinguishes these four quantities from the primal capacity slacks s_1, s_2.

Replace zero complementarity by a small positive product

Recall LP optimality: primal feasibility, dual feasibility, and complementary slackness. With $r = c - A^\top p$, the last condition is $x_j r_j = 0$.

At an LP optimum	On the central path
$Ax = b,\ x \geq 0$	$Ax = b,\ x > 0$
$A^\top p + r = c,\ r \geq 0$	$A^\top p + r = c,\ r > 0$
$x_j r_j = 0$	$x_j r_j = \mu \quad (j = 1, \dots, n)$

> The positive products keep both x and r away from zero. Lowering μ approaches the complementary-slackness equations.

Appendix: derive these equations from the standard-form barrier.

The duality gap measures a certified objective error

Suppose $Ax = b,\ x \geq 0$ and $A^\top p + r = c,\ r \geq 0$. Then:

$$c^\top x - b^\top p = x^\top (c - A^\top p) = x^\top r.$$

$$b^\top p \leq f^* \leq c^\top x, \qquad 0 \leq c^\top x - f^* \leq x^\top r.$$

On the exact central path each $x_j r_j = \mu$, so the gap is $n\mu$. Our standard-form example has four variables: its gap is 4μ.

This is a bound on the primal objective error, not a claim that the error equals 4μ. The certificate requires both points to be feasible.

A small gap brackets the unknown optimum

Our LP: minimize $f = -2u - v$, subject to $u + v \le 4,\ u \le 3,\ u, v \ge 0$.

At $\mu = 0.1$, $(u, v) \approx (2.89318, 1.01579)$, so $f \approx -6.80214$. Using $p_i = -\mu / s_i$ gives $p \approx (-1.09845, -0.93612)$.

$$b^\top p = 4p_1 + 3p_2 \approx -7.20214.$$

$$-7.20214 \ \le \ f^* \ \le \ -6.80214.$$

The interval has width 4μ = 0.4. The known optimum –7 lies inside; the actual primal error is about 0.19786.

For a large LP whose optimum is unknown, feasible primal and dual points still give this bracket.

A positive starting point is a real assumption

Variable order: $x = (u, v, s_1, s_2)^\top$, where $s_1 = 4 - u - v$ and $s_2 = 3 - u$.

For our example, $x = (1, 1, 2, 2)$ is strictly primal feasible. Also $p = (-3, -2)$ gives $r = c - A^\top p = (3, 2, 3, 2) > 0$.

> Thus this example has strictly feasible starting points for both sides. A general LP may not supply such points immediately.

A feasible-start explanation assumes those points are available. Practical methods can instead start with violated equalities and reduce those violations; the next lecture defines those residuals explicitly.

A feasible region can also force some variables to zero. Such a model requires preprocessing or a method that does not assume this strict-feasibility setup.

From a geometric path to a computable algorithm

Question	Answer
Why stay inside?	Logarithms prevent a finite barrier minimum from reaching a boundary.
What is the central path?	The minimizers of the barrier problems as μ varies.
How do we compute a point?	Solve Newton equations and control the step length.
How do we certify accuracy?	Use feasible primal/dual points and their objective gap.

> Next: write one Newton step for the primal and dual variables together, compute every number, and decide when to stop.

Additional derivations and one-dimensional examples.

Appendix: derivations and additional examples

Why our barrier has one minimizer.

Standard-form barrier and its optimality conditions.

A precise Newton step-length rule.

A complete one-dimensional example.

Textbook Example 9.3: the path need not end at a vertex.

What the analytic center means.

Why the barrier has a unique minimum in our polygon

At any strictly feasible point (u, v), take a nonzero direction $d = (d_u, d_v)$. The Hessian satisfies

$$d^\top H d = \mu \left[\frac{d_u^2}{u^2} + \frac{d_v^2}{v^2} + \frac{(d_u + d_v)^2}{s_1^2} + \frac{d_u^2}{s_2^2} \right] > 0.$$

Thus the barrier objective is strictly convex. Its value tends to $+\infty$ at every boundary of the bounded polygon. A minimum exists inside, and strict convexity makes it unique.

> A zero gradient therefore identifies the global minimum for this barrier problem.

Derive the standard-form central-path equations

For $\min c^\top x - \mu \sum_j \log x_j$ subject to $Ax = b, x > 0$, define

$$L(x, p) = c^\top x - \mu \sum_j \log x_j + p^\top (b - Ax).$$

Differentiating with respect to each x_j and setting the derivative to zero gives:

$$c_j - \mu / x_j - A_j^\top p = 0.$$

Set $r_j = c_j - A_j^\top p$. Then $r_j = \mu / x_j > 0$, or $x_j r_j = \mu$. Together with $Ax = b$, these are the displayed central-path equations.

With independent equality rows, stationarity is necessary at an interior minimizer. Convexity makes it sufficient as well.

Why these equations are sufficient for a barrier minimum

Let $x > 0$ satisfy $Ax = b$ and $\nabla F_\mu(x) = A^\top p$. For any other feasible $y > 0$, convexity gives

$$F_\mu(y) \geq F_\mu(x) + \nabla F_\mu(x)^\top (y - x) = F_\mu(x) + p^\top (Ay - Ax) = F_\mu(x).$$

> Therefore x minimizes the barrier objective over Ax = b. Its strictly positive diagonal Hessian in standard form makes the minimizer unique.

This argument concerns a linear equality constraint. Convexity of an objective alone would not give the same result for arbitrary nonlinear equalities.

A precise step-length rule for the fixed-μ Newton solve

Write the standard-form change as $d_x = (d_u, d_v, -d_u - d_v, -d_u)$. Compute

$$\alpha_{\max} = \min_{j:(d_x)_j<0} \frac{-x_j}{(d_x)_j}.$$

An empty minimum is $+\infty$. Start with $\alpha = \min(1, 0.9\alpha_{\max})$, so every component remains positive.

> While $F_\mu(x + \alpha d_x) > F_\mu(x) + 10^{-4}\alpha \nabla F_\mu(x)^\top d_x$, replace α by $\alpha/2$.

This is backtracking: repeatedly shorten a proposed step until it gives sufficient decrease. The objective and μ stay fixed during this test.

One variable: solve a bounded LP through barriers

$$\min_{0\le t\le 1} t, \qquad \phi_\mu(t) = t - \mu\log t - \mu\log(1-t).$$

$$\phi_\mu'(t) = 1 - \frac{\mu}{t} + \frac{\mu}{1-t} = 0 \quad\Longrightarrow\quad t^2 - (1+2\mu)t + \mu = 0.$$

$$t(\mu) = \frac{1 + 2\mu - \sqrt{1+4\mu^2}}{2}.$$

Use the smaller root, which lies between 0 and 1. For μ = 1, 0.1, 0.01, the minimizers are approximately 0.38197, 0.09010, 0.00990.

The original LP optimum is t = 0; every positive-μ barrier minimum is strictly between the endpoints.

One variable: a full Newton step can cross the boundary

For $\mu = 0.1$, start at $t = 0.5$. The second derivative is $\phi''_\mu(t) = \mu/t^2 + \mu/(1-t)^2$.

$$\phi'(0.5) = 1, \qquad \phi''(0.5) = 0.8, \qquad d = -1/0.8 = -1.25.$$

The full step gives –0.75, outside the domain. Taking α = 0.32 instead gives:

$$t_{\text{new}} = 0.5 + 0.32(-1.25) = 0.1.$$

The next direction is approximately –0.01098, moving near the exact barrier minimizer 0.09010. Positivity and objective decrease must be checked at each step.

Example 9.3: derive a second central path

$$\min x_2$$
$$x_1 + x_2 + x_3 = 1, \quad x \geq 0.$$

Add $-\mu \sum_{j=1}^{3} \log x_j$. Substitute $x_3 = 1 - x_1 - x_2$, then differentiate:

$$-\mu/x_1 + \mu/x_3 = 0,$$
$$1 - \mu/x_2 + \mu/x_3 = 0.$$

> The first equation gives $x_1 = x_3$. Together with the equality constraint, this means $x_1 = x_3 = (1 - x_2)/2$.

Substitute this expression into the second equation to solve for x_2.

The limit need not be a vertex

For $\min x_2$ subject to $x_1 + x_2 + x_3 = 1, x \geq 0$, the preceding equations give

$$x_2^2 - (1 + 3\mu)x_2 + \mu = 0.$$

$$x_2 = \frac{1 + 3\mu - \sqrt{1 + 2\mu + 9\mu^2}}{2}.$$

$$x_1 = x_3 = \frac{1 - x_2}{2}.$$

As $\mu \downarrow 0$, the point approaches $(1/2, 0, 1/2)$. This is optimal, but it is not a vertex. The path lies on the straight line $x_1 = x_3$.

The analytic center maximizes the product of slacks

For a bounded standard-form feasible set containing a strictly positive point, the **analytic center** minimizes $-\sum_j \log x_j$ subject to $Ax = b, x > 0$. Equivalently, it maximizes $\prod_j x_j$.

Since $F_\mu/\mu = (c^\top x)/\mu - \sum_j \log x_j$, the central path approaches this center as $\mu \to \infty$.

$$u_{\text{center}} = \frac{17 - \sqrt{97}}{8}, \qquad v_{\text{center}} = \frac{15 + \sqrt{97}}{16}.$$

For our polygon this is approximately (0.89389,1.55305). It is defined by the logarithmic objective, not by equal Euclidean distances to the boundaries.

A large but finite μ still includes the linear objective. The μ = 10 point is not exactly the analytic center.

Conditions behind the standard central-path description

A sufficient standard setup is: A has independent rows; there is x > 0 with Ax = b; and there is an unrestricted p with r = c – $A^\top p > 0$.

For any such fixed dual point, on $Ax = b$, $c^\top x = b^\top p + r^\top x$. Positive components of r make the linear term grow faster than the logarithms along unbounded feasible sequences.

At a finite boundary the barrier tends to +∞. Together these facts ensure an interior minimizer; strict convexity makes it unique.

> Our main example satisfies these assumptions. Do not assume they hold for every LP without checking or preprocessing.