Network Simplex II: Bounds and Starting Bases

The graph pivot from Network Simplex I assumed that arcs had no upper capacities and that a feasible tree was already available.

Question	Today's extension
An increasing arc fills up	Include upper capacities in the ratio test
The step length is zero	Distinguish flows from the basis tree
No starting tree is obvious	Construct an auxiliary Phase I network
Why do whole flows appear?	Use integer data and incidence-matrix structure

Complete examples come before the general rules; longer proofs and additional cases are in the appendix.

Recall the uncapacitated pivot

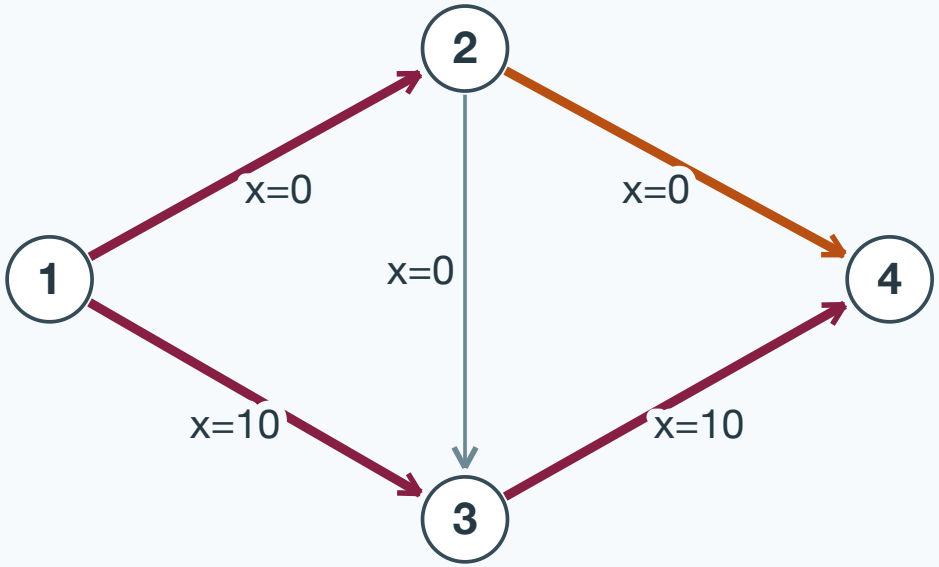
Ship 10 units from 1 to 4. Arc order $(12, 13, 23, 24, 34)$, costs $(2, 3, 2, 1, 4)$, balances $(10, 0, 0, -10)$.

Initial tree {12, 13, 34}: x = (0, 10, 0, 0, 10), cost 70.

Enter arc 24. Increasing it by θ increases x_{12} and decreases x_{13} and x_{34}.

$$x(\theta) = (\theta, 10 - \theta, 0, \theta, 10 - \theta).$$

Without upper bounds, θ = 10 and the final cost is 30. What if arc 12 can carry only 4 units?



Arrow: original arc direction. Labels: current flow. Thick maroon arcs form the basis tree.

New data: every arc now has a capacity

Ship 10 units from 1 to 4. Arc order $(12, 13, 23, 24, 34)$, costs $(2, 3, 2, 1, 4)$, balances $(10, 0, 0, -10)$.

Arc	12	13	23	24	34
Unit cost	2	3	2	1	4
Upper capacity u	4	10	10	6	10
Initial flow	0	10	0	0	10

$$\min c^\top x \quad \text{s.t. } Ax = b, \quad 0 \le x \le u.$$

With explicit upper bounds, a **nonbasic flow is fixed at either 0 or its upper capacity**. The basic tree flows are solved from the node balances.

The initial flow is feasible. A move stops at zero or capacity. An arc with capacity 0 is fixed at zero and can be removed.

Check room to increase and room to decrease

Ship 10 units from 1 to 4. Arc order $(12, 13, 23, 24, 34)$, costs $(2, 3, 2, 1, 4)$, balances $(10, 0, 0, -10)$.

Cycle arc	New flow	Bound on θ
12	$\theta \le 4$	4
24	$\theta \le 6$	6
13	$10 - \theta \ge 0$	10
34	$10 - \theta \ge 0$	10

$$\theta^* = \min\{4, 6, 10, 10\} = 4.$$

Arc 12 reaches capacity and leaves the tree. Fix its nonbasic flow at x_{12} **= 4**; those 4 units still travel on arc 12. Arc 24 becomes basic.

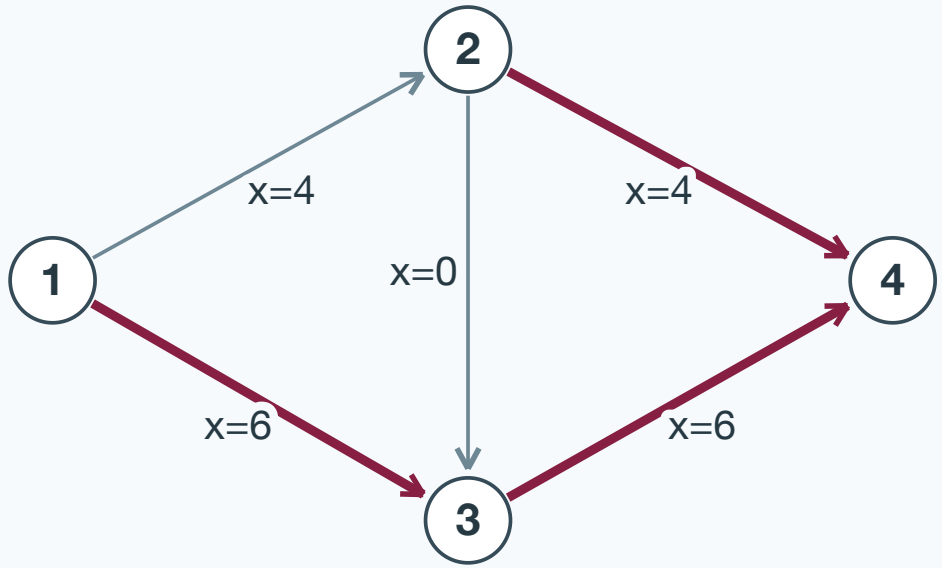
Nonbasic means fixed at a bound

After the pivot, the tree is T = {13, 24, 34}. Arc 12 is outside the tree, but still carries its full capacity of 4.

Status	Arcs and current flows
Basic tree	$x_{13} = 6,\ x_{24} = 4,\ x_{34} = 6$
Nonbasic at zero	$x_{23} = 0$
Nonbasic at capacity	$x_{12} = u_{12} = 4$

Fix the two nonbasic values, then solve the three tree flows from the balances.

The total cost is $f = 54$.



Arc labels: flows. Thick arcs: basic tree. Capacities in arc order (12, 13, 23, 24, 34): (4, 10, 10, 6, 10).

> The chosen tree determines which flows are basic. A basic flow may also equal zero or its capacity.

At the upper bound, unused capacity is zero

Arc 12 has capacity 4. Define its **unused capacity** $s_{12} = 4 - x_{12} \geq 0$. Then $x_{12} = 4$ means $s_{12} = 0$.

Substitute $x_{12} = 4 - s_{12}$ into the node balances for tree $\{13, 24, 34\}$:

$$\begin{aligned} x_{13} &= 10 - x_{12} = 6 + s_{12}, \\ x_{24} &= x_{12} - x_{23} = 4 - s_{12} - x_{23}, \\ x_{34} &= x_{13} + x_{23} = 6 + s_{12} + x_{23}. \end{aligned}$$

Substitute these flows into the original cost $f = 2x_{12} + 3x_{13} + 2x_{23} + x_{24} + 4x_{34}$:

$$f = 54 + 4s_{12} + 5x_{23}.$$

> Set the nonbasic variables $s_{12} = x_{23} = 0$, as in our earlier dictionaries. This gives $x_{12} = 4$. Keep all original capacity bounds.

The matrix form includes fixed upper-bound flows

Let A be the incidence matrix with one redundant row deleted, and B its tree columns. Partition arcs into tree T, lower-bound nonbasic L, and upper-bound nonbasic U: $x_L = 0$, $x_U = u_U$.

$$Bx_T + A_U u_U = b \quad \Rightarrow \quad x_T = B^{-1}(b - A_U u_U).$$

Delete node 4's balance row. Here $U = \{12\}$ and $u_{12} = 4$, so

$$b - A_{12}(4) = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}.$$

Solve the tree balances: x_{13} = 6, x_{24} = 4, and -x_{13} + x_{34} = 0 gives x_{34} = 6.

Why the capacity-4 flow is optimal

Ship 10 units from 1 to 4. Arc order $(12, 13, 23, 24, 34)$, costs $(2, 3, 2, 1, 4)$, balances $(10, 0, 0, -10)$.

For tree $\{13, 24, 34\}$, $p = (0, -6, -3, -7)$. Nonbasic flows: $x_{12} = 4$ at capacity and $x_{23} = 0$.

Arc	Reduced cost	Change of size $\theta \geq 0$	Cost change
12	$2 - 0 - 6 = -4$	$x_{12} = 4 - \theta$	$+4\theta$
23	$2 - (-6) - 3 = 5$	$x_{23} = \theta$	$+5\theta$

$$f = 54 + 4s_{12} + 5x_{23}, \qquad s_{12} = 4 - x_{12} \geq 0.$$

Every feasible flow has $s_{12} \geq 0$ and $x_{23} \geq 0$, so $f \geq 54$. The current flow $(4, 6, 0, 4, 6)$, in the arc order above, attains 54 and is optimal.

Appendix: the complete dual and the optimality identity.

A second capacity case: the entering arc fills first

Restart from the same initial flow (0, 10, 0, 0, 10). Change the capacities to (10, 10, 10, 6, 10): only arc 12's capacity is larger than before.

$$x(\theta) = (\theta, 10 - \theta, 0, \theta, 10 - \theta), \qquad \theta^* = 6.$$

Arc 24 reaches its own upper bound before any tree arc reaches a bound.

> Keep the same tree {12, 13, 34}. Move nonbasic arc 24 from its lower-bound set to its upper-bound set. This is called a **bound flip**.

$$x = (6, 4, 0, 6, 4), \qquad f = 70 - 4(6) = 46.$$

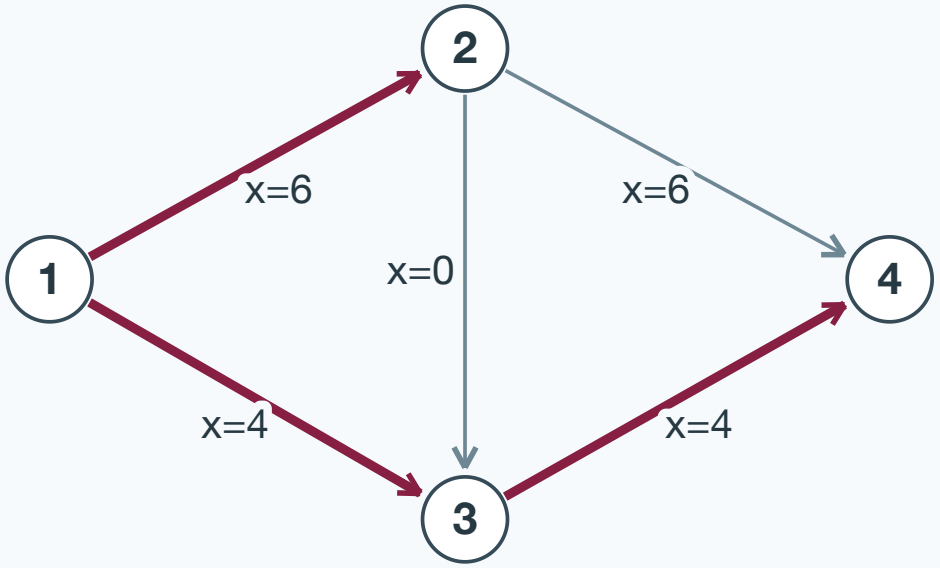
Your turn: verify the bound-flip answer

In arc order (12, 13, 23, 24, 34), use costs (2, 3, 2, 1, 4) and capacities (10, 10, 10, 6, 10). Node balances are (10, 0, 0, -10).

Work for about 4 minutes.

1. Check all balances and capacities for x = (6, 4, 0, 6, 4).

2. The unchanged tree gives p = (0, -2, -3, -7). Compute the reduced costs of nonbasic arcs 23 and 24. Which way can each flow change, and would that lower cost?



Arc labels: flows. Thick arcs: basic tree. Capacities in arc order (12, 13, 23, 24, 34): (10, 10, 10, 6, 10).

A bound flip can finish the problem

Flow (6, 4, 0, 6, 4), in arc order (12, 13, 23, 24, 34), satisfies all capacities and node balances:

$$6 + 4 = 10, \quad -6 + 0 + 6 = 0, \quad -4 - 0 + 4 = 0, \quad -6 - 4 = -10.$$

Nonbasic arc 23 is at zero with reduced cost 1: increasing it raises cost. Nonbasic arc 24 is at capacity 6 with reduced cost -4: decreasing it raises cost.

Define unused capacity $s_{24} = 6 - x_{24} \geq 0$. In the unchanged tree dictionary,

$$f = 70 + x_{23} - 4x_{24} = 46 + x_{23} + 4s_{24} \geq 46.$$

> The current flow has $x_{23} = s_{24} = 0$, so its cost 46 is optimal. This improving step changed a nonbasic flow's bound without changing the tree.

Both capacity examples are complete. Next, summarize the entering rules and step limits.

Summary: choose the allowed improving direction

For minimization, compute $p_i - p_j = c_{ij}$ on the tree and $\bar{c}_{ij} = c_{ij} - p_i + p_j$ on all arcs.

Nonbasic flow	Allowed direction	Candidate to enter
At lower bound 0	Increase flow	$\bar{c}_{ij} < 0$
At upper bound u_{ij}	Decrease flow	$\bar{c}_{ij} > 0$

If $\bar{c}_e \geq 0$ on L and $\bar{c}_e \leq 0$ on U, the current feasible tree flow is optimal. Tree reduced costs are zero. A chosen entering arc may still have step length zero.

An upper-bound variable with negative reduced cost is not an improving candidate: increasing it is forbidden.

The bounded cycle ratio test

Choose a lower-bound arc to increase or an upper-bound arc to decrease. Orient the cycle in that chosen change direction.

$$\theta^* = \min\left\{ \min_{e \text{ increasing}} (u_e - x_e), \quad \min_{e \text{ decreasing}} x_e \right\}.$$

Include the entering arc in this test. An infinite capacity contributes no finite restriction.

> If a tree arc reaches a bound, exchange it with the entering arc. If only the entering arc reaches its opposite bound, perform a bound flip. With tied blockers, more than one valid update may exist.

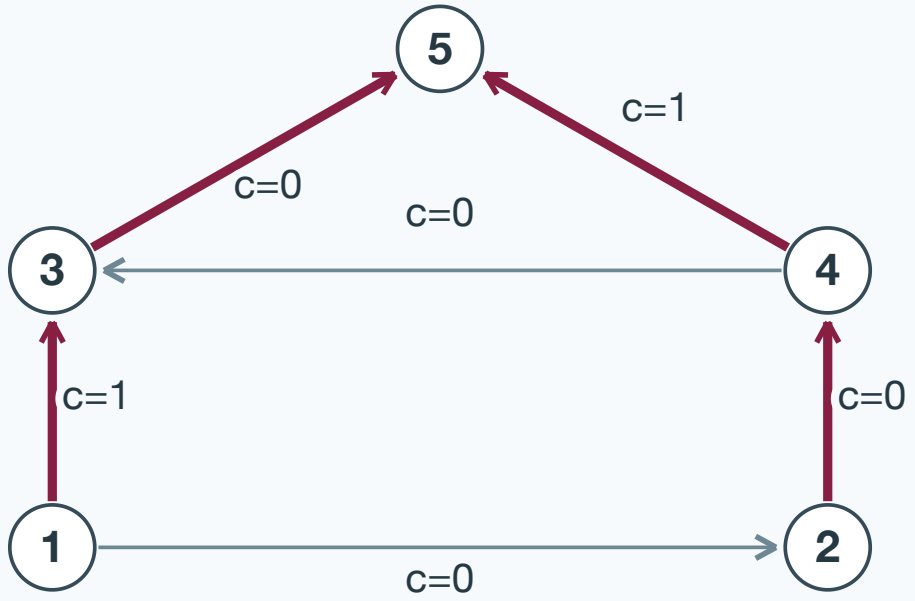
The cost change is θ times the reduced cost when increasing, and minus θ times it when decreasing. If no cycle bound gives a finite limit on θ, this improving direction proves that the objective is unbounded below.

A new example: a basis change without movement

Send one unit from node 1 to node 5. Other node balances are zero. All flows are nonnegative, with no upper bounds.

Only arcs 13 and 45 have cost 1; the other four arcs have cost 0. Thus minimize f = $x_{13} + x_{45}$.

> Initial tree {13, 24, 35, 45}. Its flows are $x_{13} = x_{35} = 1$, $x_{24} = x_{45} = 0$. The cost is 1.



Arrow: original arc direction. Labels: cost per unit. Node balances are stated beside the graph.

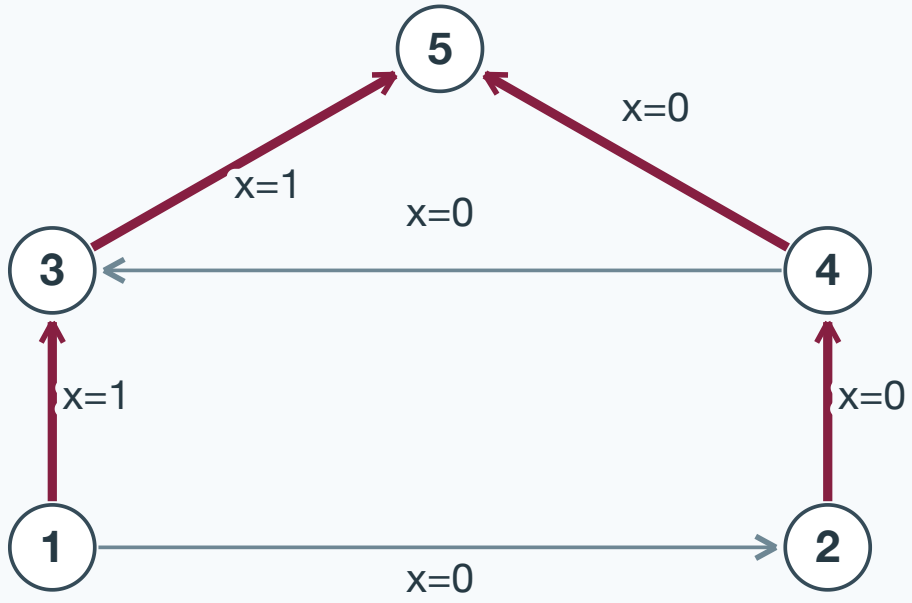
Pivot 1: an improving direction with a zero step

Set $p_5 = 0$. The tree gives $p = (1, 1, 0, 1, 0)$, so $\bar{c}_{43} = 0 - 1 + 0 = -1$. Enter arc 43.

Cycle $4 \to 3 \to 5 \to 4$ increases x_{43} and x_{35}, but decreases x_{45}.

$$x_{45} = 0 - \theta \geq 0 \quad \Rightarrow \quad \theta^* = 0.$$

> Arc 45 leaves. Flows and objective do not change, but the tree becomes {13, 24, 35, 43}.



Send one unit from 1 to 5. Labels: flows. Thick maroon arcs: the current basis tree.

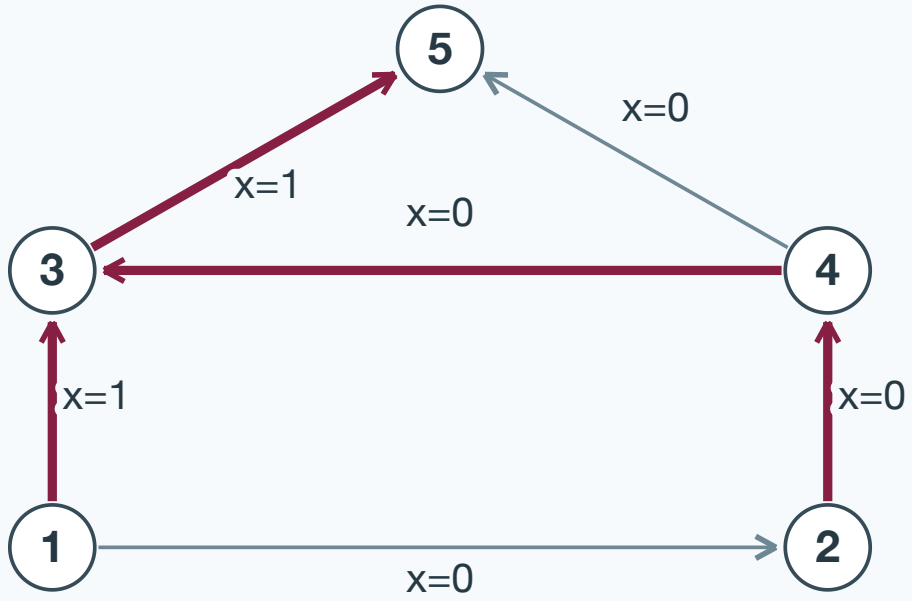
Pivot 2: the new tree reveals a positive move

The new tree gives $p = (1, 0, 0, 0, 0)$. Now $\bar{c}_{12} = 0 - 1 + 0 = -1$. Enter arc 12.

Cycle $1 \to 2 \to 4 \to 3 \to 1$ increases x_{12}, x_{24}, x_{43} and decreases x_{13}.

$$\theta^* = x_{13} = 1. \quad \text{Arc 13 leaves.}$$

The first zero-length pivot changed the basis information used to find this direction.



Send one unit from 1 to 5. Labels: flows. Thick maroon arcs: the current basis tree.

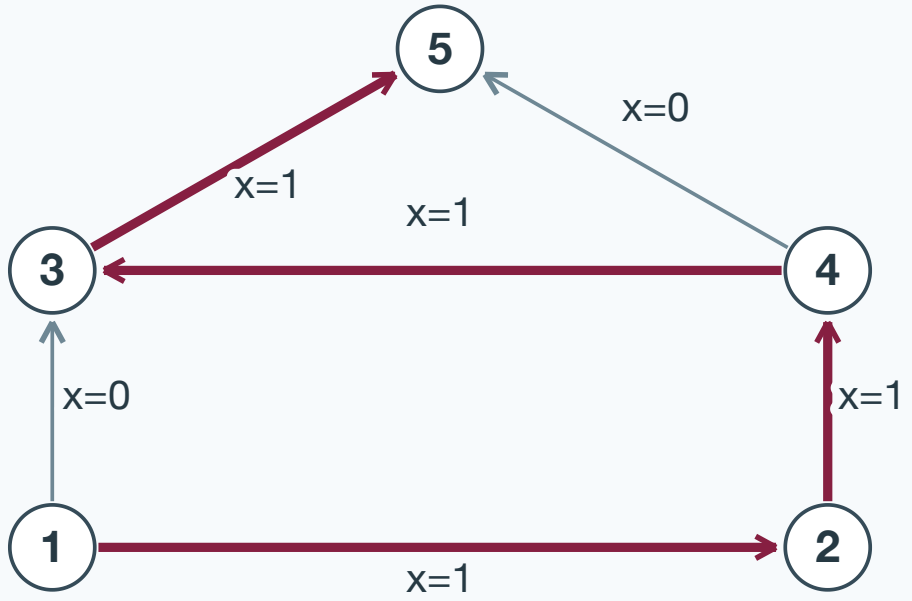
The second pivot reaches cost zero

Final positive flows: $x_{12} = x_{24} = x_{43} = x_{35} = 1$. The unused flows x_{13} and x_{45} are zero.

$$f = x_{13} + x_{45} = 0.$$

All original arc costs are nonnegative, so no feasible flow can cost below zero. Equivalently, p = 0 is dual feasible and has value zero.

> A zero step does not mean that the current flow is optimal or that the algorithm should stop.



Send one unit from 1 to 5. Labels: flows. Thick maroon arcs: the current basis tree.

Zero steps require a rule that prevents cycling

As in ordinary simplex, several bases can represent the same feasible flow. A sequence of zero-length pivots can revisit a basis.

> A proved anticycling rule is still needed for a finite-termination guarantee. Bland’s rule or a lexicographic rule can be applied to the standard-form LP.

Choosing an arbitrary tied arc is acceptable for a displayed finite calculation, but it is not by itself a general anticycling theorem.

Some network implementations use a specially rooted tree and a compatible tie rule. The next page defines the tree property; it does not replace the complete pivot rule.

A rooted-tree property used in network implementations

Choose a root r. A feasible tree is **strongly feasible toward r** if each node's tree path to r follows positive-capacity residual arcs: increase flow in an original arrow's direction, or decrease it in the reverse direction.

Tree arc at a bound	Required arrow direction along the path to r
Flow 0	Toward r, so the flow can increase
Flow u > 0	Away from r, so the flow can decrease
Strictly between bounds	Either direction works

> This tests the available capacity along each path; sending extra flow would change its endpoint balances. It does not say that arbitrary supplies are feasible or that every pivot has positive length.

Appendix: check this property in the zero-step example.

Where does the first feasible tree come from?

A cheapest-looking spanning tree can give negative flows or violate capacities. We need a systematic start, just as in ordinary simplex.

> Use Phase I: add artificial arcs that make the balance equations easy to satisfy. Minimize total artificial flow, temporarily ignoring the original costs.

If the minimum artificial flow is zero, remove artificial arcs from the basis and restore the original objective. If it is positive, the original flow problem is infeasible.

The artificial arcs represent auxiliary variables, not real shipping routes.

Add an artificial root and satisfy every balance

Assume $\sum_i b_i = 0$ and lower bounds are zero. Add a new root r with balance 0.

Original node balance	Artificial arc	Initial artificial flow
b_i > 0	i → r	b_i
b_i < 0	r → i	-b_i
b_i = 0	i → r	0

Set every original arc flow to zero. The artificial star is a spanning tree on the enlarged network, and all its flows are nonnegative.

$$\min w = \sum_{\text{artificial arcs } e} x_e.$$

Artificial arcs have no upper bounds. Keep the original arcs' capacity constraints; their Phase I costs are zero.

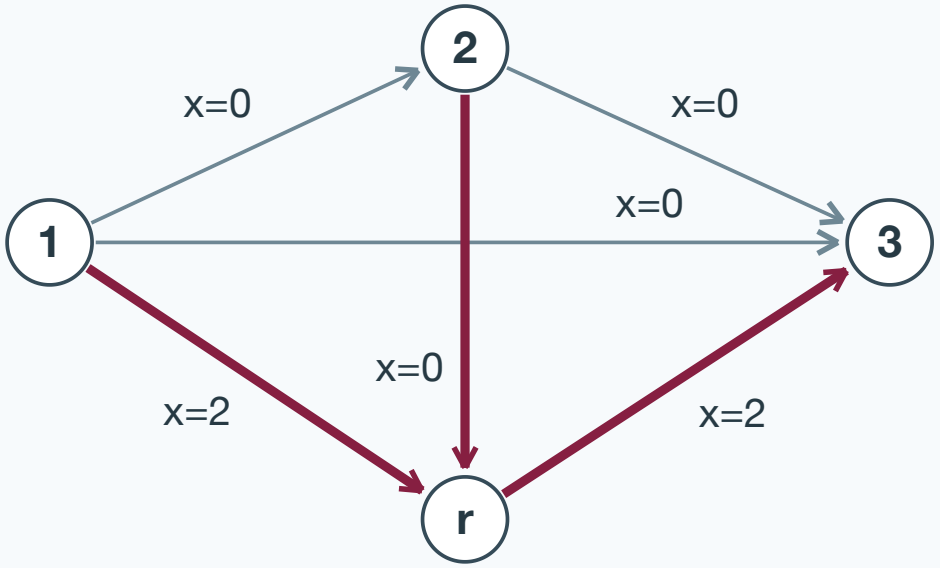
Phase I example: two units must travel from 1 to 3

Original arcs: 12, 23, 13. Original costs: 2, 1, 5; no upper bounds. Original balances: (2, 0, -2).

Add artificial arcs 1 → r, 2 → r, r → 3. Initial artificial flows are (2, 0, 2), so w = 4.

> For Phase I, every original arc has cost 0 and every artificial arc has cost 1. The original transportation costs wait until Phase II.

The initial tree consists of the three artificial arcs. All original flows start at zero.



Auxiliary network. Arcs touching r are artificial. Labels: flows; thick arcs: current basis.

Use the auxiliary costs to choose the first entering arc

Set the artificial root's dual variable to $p_r = 0$. The three basic artificial arcs have Phase I cost 1.

$$\begin{aligned} p_1 - p_r &= 1 \quad \Rightarrow p_1 = 1, \\ p_2 - p_r &= 1 \quad \Rightarrow p_2 = 1, \\ p_r - p_3 &= 1 \quad \Rightarrow p_3 = -1. \end{aligned}$$

Original arc	Phase I reduced cost
12	$0 - 1 + 1 = 0$
23	$0 - 1 + (-1) = -2$
13	$0 - 1 + (-1) = -2$

> Choose arc 23 among the two tied improving arcs. Use auxiliary reduced costs while minimizing w.

Phase I pivot 1: remove a zero artificial arc

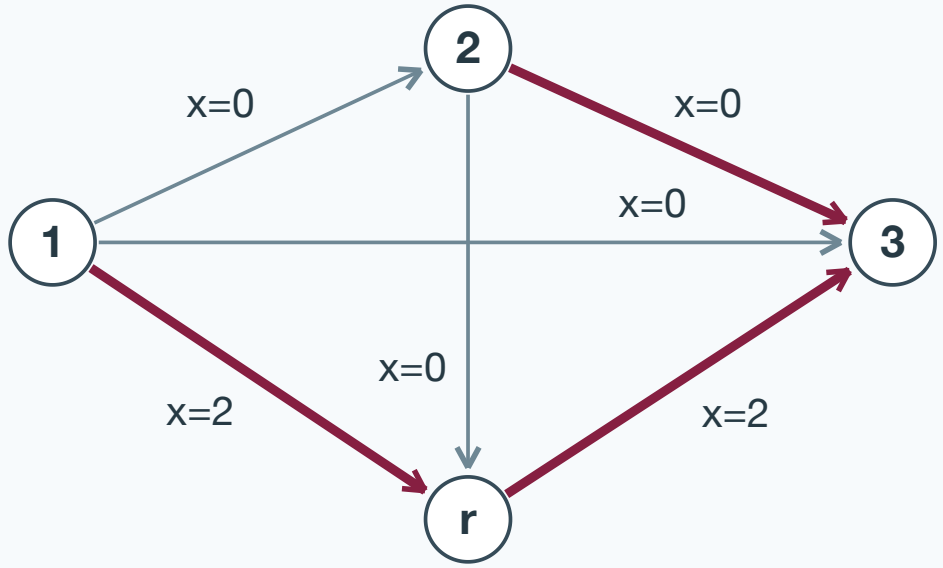
Cycle 2 → 3 → r → 2 increases x_{23}, decreases x_{r3}, and decreases x_{2r}.

$$\theta^* = \min\{x_{r3}, x_{2r}\} = \min\{2, 0\} = 0.$$

Arc 2r leaves; arc 23 becomes basic at zero. The new basis dual values are p = (1, -1, -1, 0).

$$\bar{c}_{12} = 0 - 1 + (-1) = -2.$$

> Artificial flow is still 4, but the new tree allows arc 12 to enter.



Auxiliary network. Arcs touching r are artificial. Labels: flows; thick arcs: current basis.

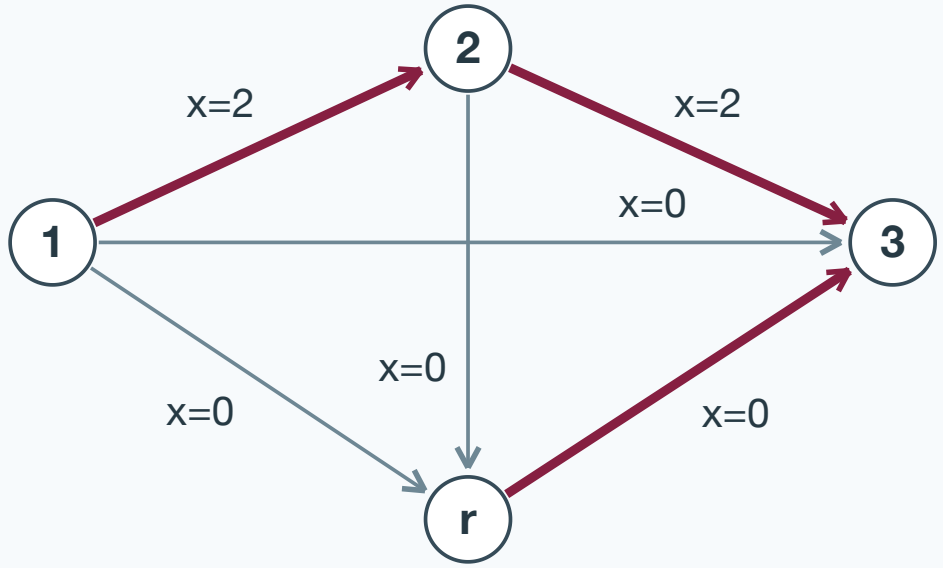
Phase I pivot 2: eliminate the positive artificial flows

Enter 12 and follow 1 → 2 → 3 → r → 1. Increase x_{12} and x_{23}; decrease x_{r3} and x_{1r}.

$$\theta^* = \min\{2, 2\} = 2, \qquad w = 4 - 2(2) = 0.$$

Choose 1r to leave. The tree is now {12, 23, r3}, with flows (2, 2, 0).

> The only remaining artificial tree arc is r3 at zero. Delete it and the artificial root. The original arcs {12, 23} form a feasible tree.



Auxiliary network. Arcs touching r are artificial. Labels: flows; thick arcs: current basis.

Restore the original costs and check optimality

Original problem: send 2 units from 1 to 3 on arcs 12, 23, 13, with costs 2, 1, 5. The feasible tree {12, 23} carries $x_{12} = x_{23} = 2$; $x_{13} = 0$.

$$f = 2(2) + 1(2) = 6.$$

Set $p_1 = 0$. Tree equations give $p_2 = -2$, $p_3 = -3$.

$$\bar{c}_{13} = 5 - 0 - 3 = 2 \geq 0, \qquad b^\top p = 2(0) - 2(-3) = 6.$$

> The original flow is already optimal. A zero Phase I objective proves feasibility; the original reduced-cost test proves optimality for the original costs.

Your turn: why have all the flows been integers?

Work for about 4 minutes. In a tree with integer supplies and demands, start at a leaf. Its balance determines the only incident tree flow.

1. Why must that flow be an integer?
2. After removing the leaf, why is the remaining node balance still an integer?
3. With integer capacities, why is a nondegenerate cycle step also an integer?

Here “integer” means a whole number. The LP still allows real-valued flows; we are examining a property of its basic solutions.

Integer data give integer basic feasible flows

A leaf equation fixes its one tree flow to an integer balance, possibly after a sign change. Removing the leaf updates the neighboring balance by an integer. Repeat.

For upper-bound nonbasic arcs, subtract their fixed integer flows from the node balances first. The remaining tree problem still has integer balances.

> **Integrality theorem.** If node balances and every finite capacity are integers, every basic feasible flow is integral. Upper bounds may also be infinite. If the minimum-cost flow LP has a finite attained optimum, an integral optimal flow exists.

Costs need not be integers for this existence statement. Not every optimal flow must be integral: convex combinations of different optimal whole flows can be fractional.

The matrix reason: total unimodularity

A matrix is **totally unimodular (TU)** if every square submatrix has determinant 0, +1, or -1.

> A directed node–arc incidence matrix is TU. Deleting a redundant row, changing a column sign, or adjoining bound rows from an identity matrix preserves TU.

An invertible basis made from such a matrix has determinant ±1. Its inverse has integer entries. Multiplying by an integer right-hand side gives integer basic values.

This explains integrality for network flow, maximum flow, and assignment LPs without imposing separate integer-variable restrictions.

Appendix: determinant proof and the precise LP integrality argument.

What you should be able to do now

Task	Evidence to write down
A bounded pivot	Cycle signs, every remaining capacity, leaving arc or bound flip
A zero-length pivot	Unchanged flows; changed tree and reduced costs
Phase I	Artificial tree, objective w, and the conclusion when $w = 0$ or $w > 0$
Integral optimum	Integer data + incidence structure, not an integer constraint

> Network simplex is ordinary simplex using graph structure. Flow feasibility and reduced-cost optimality remain separate conditions.

Appendix: proofs, lower bounds, infeasibility and further practice.

Appendix: complete arguments and extensions

Bounded optimality: primal and dual

A worked upper-bound entering variable

Positive lower bounds

Strong feasibility and zero steps

Phase I and removal of artificial arcs

Incidence matrices are totally unimodular

The dual includes upper-capacity multipliers

For finite upper capacities u, the primal and dual are:

$$\begin{aligned} \min\ & c^\top x \\ \text{s.t. } & Ax = b, \\ & 0 \le x \le u, \end{aligned}$$

$$\begin{aligned} \max\ & b^\top p - u^\top q \\ \text{s.t. } & A^\top p - q \le c, \\ & q \ge 0, \quad p \text{ unrestricted.} \end{aligned}$$

For a feasible tree satisfying the sign test, let $q_e = 0$ on T and L, and $q_e = -\bar{c}_e \ge 0$ on U. This makes the dual feasible.

$$c^\top x = p^\top b + \sum_{e \in U} \bar{c}_e u_e = p^\top b - u^\top q.$$

For the cost-54 example, p = (0, -6, -3, -7), q_{12} = 4, other q = 0: the dual value is 70 - 4(4) = 54.

The bounded sign test compares against every feasible flow

Let x be the current feasible tree flow and y any other feasible flow. Because $A(y-x)=0$,

$$c^\top (y-x) = \bar{c}^\top (y-x) = \sum_{e \in L} \bar{c}_e y_e + \sum_{e \in U} \bar{c}_e (y_e - u_e).$$

On L, both factors are nonnegative. On U, both factors are nonpositive. Every product is nonnegative; tree terms vanish because their reduced costs are zero.

Therefore $c^\top y \geq c^\top x$. This proves sufficiency of the sign test even with zero or capacity-valued basic flows.

An upper-bound flow can enter by decreasing

Return to the capacity-4 example: x = (4, 6, 0, 4, 6), tree {13, 24, 34}, capacities (4, 10, 10, 6, 10).

Change only c_{12} from 2 to 8.

Basic costs are unchanged, so p = (0, -6, -3, -7). Now the upper-bound arc 12 has reduced cost 8 - 0 - 6 = 2 > 0.

$$x(\theta) = (4-\theta,\ 6+\theta,\ 0,\ 4-\theta,\ 6+\theta).$$

All four changed arcs reach bounds at θ = 4. Choose arc 24 to leave at zero, so the tree becomes {12, 13, 34}.

$$x = (0, 10, 0, 0, 10), \quad f_{\text{old}} = 78, \quad f_{\text{new}} = 78 - 2(4) = 70.$$

Check the changed-cost example after the pivot

Costs are now (8, 3, 2, 1, 4). The final tree {12, 13, 34} has flows (0, 10, 10); the two other arc flows are zero.

$$p = (0, -8, -3, -7), \qquad \bar{c}_{23} = 2 + 8 - 3 = 7, \quad \bar{c}_{24} = 1 + 8 - 7 = 2.$$

All nonbasic arcs are at their lower bounds with nonnegative reduced costs. The basis is degenerate because basic flows lie at bounds; the optimality proof still applies.

$$c^\top x = 70 = b^\top p.$$

> The original $c_{12} = 2$ example and this changed $c_{12} = 8$ example are different optimization problems. Recompute reduced costs whenever objective coefficients change.

Remove positive lower bounds by a variable shift

For a problem with $\ell \le x \le u$, define $y = x - \ell$. Then

$$Ay = b - A\ell, \qquad 0 \le y \le u - \ell, \qquad c^\top x = c^\top \ell + c^\top y.$$

Thus the same zero-lower-bound algorithm applies to y, with adjusted balances and capacities. Add ℓ back to recover x.

> Example: arc $1 \to 2$ must carry at least 3 and at most 8. Preassign 3 units; its remaining capacity is 5. Reduce node 1's required outgoing balance by 3 and increase node 2's balance by 3.

If all bounds and balances are integers, the adjusted data are still integers. Fixed arcs with ℓ = u can be removed after this adjustment.

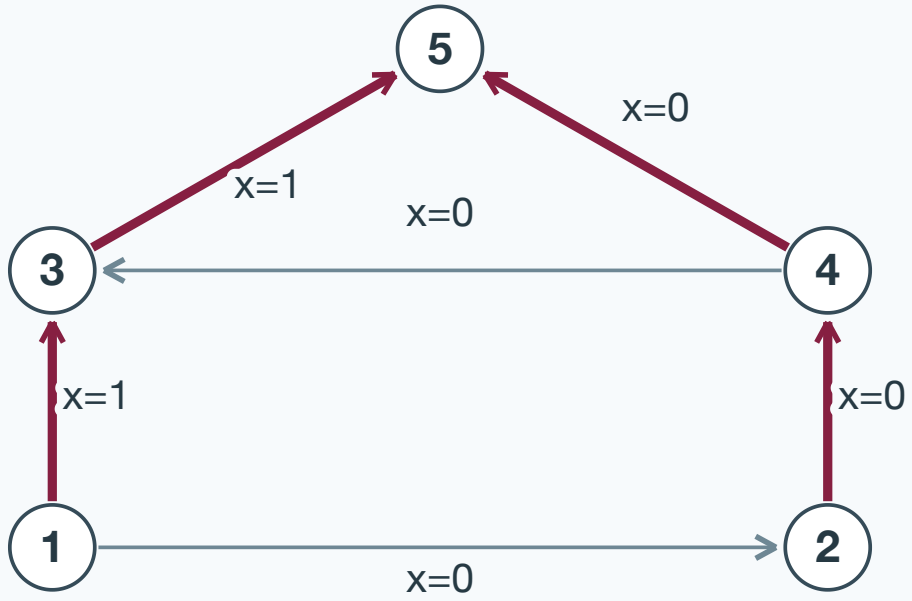
A strongly feasible tree can still take a zero step

In the initial five-node example, choose node 5 as root. Tree arcs 24 and 45 have zero flow and point toward root 5. Positive tree flows have no upper bounds.

A small additional amount can therefore travel from each node toward 5 along the tree. The tree is strongly feasible.

> Yet entering 43 decreases zero-valued x_{45} and gives θ = 0. Strong feasibility is not a statement that all improving cycle directions have positive step length.

A specialized no-cycling theorem must specify how this property is preserved when selecting a leaving arc.



Send one unit from 1 to 5. Labels: flows. Thick maroon arcs: the current basis tree.

Why the auxiliary minimum decides feasibility

Every auxiliary variable is nonnegative, so $w \geq 0$. If the original flow problem is feasible, extend any feasible original flow by setting all artificial flows to zero. This is auxiliary-feasible with $w = 0$.

Conversely, if an auxiliary feasible flow has $w = 0$, every artificial flow is zero. Its original arc flows satisfy the original balances and capacities.

> Original feasible $\Leftrightarrow$ Phase I optimum equals zero. A positive auxiliary optimum proves infeasibility.

Because the artificial star is feasible and w is bounded below, Phase I has a finite optimum. Use an anticycling rule when zero steps occur.

Zero artificial flows may still be basic

At w = 0, all artificial flows are zero. If at least two artificial arcs remain in the tree, its original tree arcs form separate components. Choose an original arc joining two of those components.

Adding that arc creates a cycle through the artificial root. Remove an artificial arc on this cycle: the basis changes but the flow does not. Repeat until only one artificial tree arc remains; then delete that arc and the root.

> Assumption for this construction: the original graph is connected after fixed arcs are eliminated. If it has separate components, solve each component separately; every component must have balanced net supply.

This is the same zero-basic-artificial-variable issue as ordinary Phase I. Do not delete a basic column before replacing it or removing a redundant equation.

A small infeasible Phase I example

Two nodes require b = (3, -3), but the only original arc 1 → 2 has capacity 2. Add artificial arcs 1 → r and r → 2.

$$x_{12} + x_{1r} = 3, \qquad x_{12} + x_{r2} = 3, \qquad 0 \le x_{12} \le 2.$$

$$w = x_{1r} + x_{r2} = 6 - 2x_{12} \ge 2.$$

The auxiliary optimum is 2, attained at x_{12} = 2 and both artificial flows equal to 1. The original shipment requirement is infeasible: only 2 of the required 3 units can pass.

Why every incidence subdeterminant is 0 or ±1

Consider any square submatrix M of a directed incidence matrix. Prove the claim by induction on its size; a 1×1 entry is 0, 1, or -1.

If a column has no nonzero entry, $\det M = 0$. If a column has exactly one nonzero entry, expand along it: the determinant is ± the determinant of a smaller submatrix.

Otherwise every column has two nonzero entries. They must be +1 and -1, so the sum of the rows is zero and $\det M = 0$.

> These cases exhaust all possibilities. Deleting rows or columns preserves TU. Adjoining a signed unit row or column preserves TU by the same determinant expansion.

From determinant ±1 to integer basic solutions

For each finite capacity, add $x_e + s_e = u_e$ with $s_e \geq 0$. Keep $Ax = b$ and $x \geq 0$.

After deleting dependent balance rows, the matrix is the reduced incidence matrix with unit rows for bounds and unit columns for slacks. It is TU.

Let B be an invertible basis and h its integer right-hand side. Replace column k of B by h to form B_k. Cramer's rule gives the kth basic value:

$$x_{B,k} = \frac{\det(B_k)}{\det B}, \qquad \det B = \pm 1.$$

The matrix B_k has integer entries, so its determinant is an integer. Dividing by +1 or -1 gives an integer basic value.

> If the standard-form LP has a finite optimum, an optimal basic feasible solution exists. Its integer basic values give an integral optimal flow.

Integer arc costs also give integer basis dual variables

Assume every original arc cost is an integer. Choose a tree and set the dual variable at its root to zero. On each tree arc, use the same basis equation as before:

$$p_i - p_j = c_{ij}.$$

Starting at the root, find each new node value by adding or subtracting an integer arc cost. Every computed dual value is therefore an integer.

> This holds for a tree's dual candidate. Feasibility still requires checking every nonbasic reduced cost, with the appropriate sign at its lower or upper bound.

For an uncapacitated problem, network simplex terminates optimally when all nonbasic reduced costs are nonnegative. Its integer dual values then give an integer optimal dual solution. Supplies need not be integers for this dual statement.

Integral basic solutions do not exclude fractional optima

There are two routes from s to t, each of total cost 1. Send one unit, with enough capacity on both routes.

Feasible optimum	First route	Second route	Cost
Whole flow A	1	0	1
Whole flow B	0	1	1
Their average	0.5	0.5	1

> The fractional average is also optimal, but it is not an extreme point of this feasible segment. The theorem guarantees an integral optimal basic solution, not that every optimum is integral.