Network Simplex I: A Pivot on a Graph

A minimum-cost flow problem is an LP. Network simplex performs the familiar simplex steps using a spanning tree.

Ordinary simplex	Network calculation
Choose basic columns	Choose a spanning tree
Solve for basic variables	Use node-balance equations
Compute reduced costs	Use one dual variable per node
Choose direction and ratio	Follow a cycle and find a blocking arc

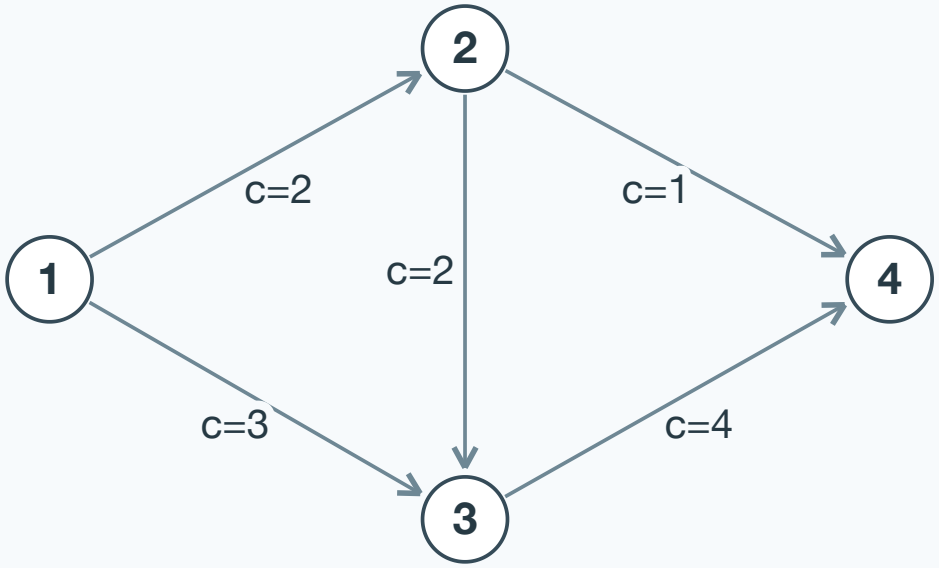
> We will compute one complete pivot in equations, a graph, and matrix notation.

A shipping problem with four locations

A depot at node 1 has 10 units to ship to a customer at node 4. Nodes 2 and 3 are transfer locations: shipments neither appear nor disappear there.

The flow x_{ij} is the amount sent on the arrow $i \to j$. Its unit cost is c_{ij}. In this first example, arcs have no upper capacity limits.

> Choose nonnegative flows to satisfy the required deliveries at minimum total transportation cost.

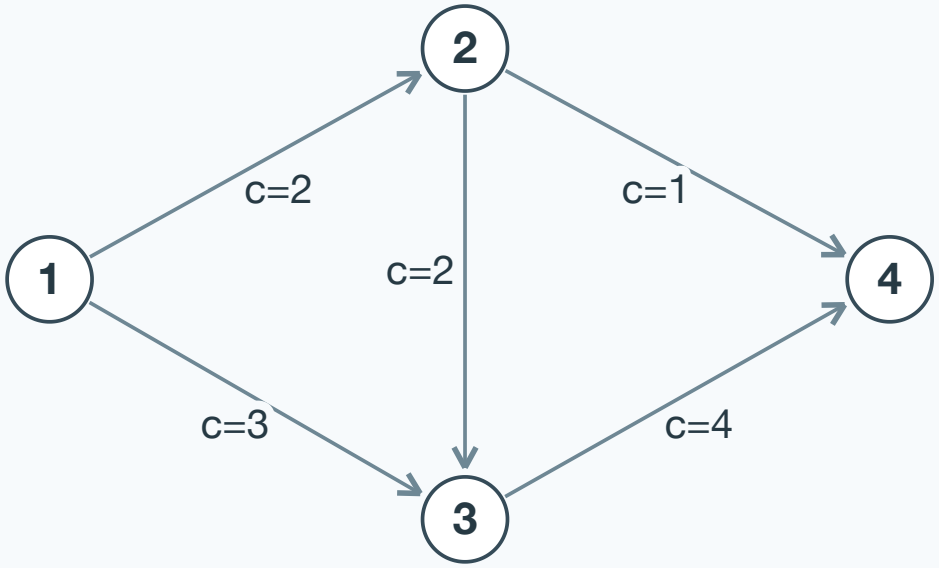


Arrow: original arc direction. Labels: cost per unit. Node balances are stated beside the graph.

Write every node balance before using a matrix

$$
\begin{aligned}
x_{12} + x_{13} &= 10 && \text{(node 1)},\\
-x_{12} + x_{23} + x_{24} &= 0 && \text{(node 2)},\\
-x_{13} - x_{23} + x_{34} &= 0 && \text{(node 3)},\\
-x_{24} - x_{34} &= -10 && \text{(node 4)}.
\end{aligned}
$$

We use outgoing flow minus incoming flow. Positive b means supply; negative b means demand.



Arrow: original arc direction. Labels: cost per unit. Node balances are stated beside the graph.

$$\min f = 2x_{12} + 3x_{13} + 2x_{23} + x_{24} + 4x_{34}, \qquad x \geq 0.$$

The incidence matrix records those same equations

Use the arc order $(12, 13, 23, 24, 34)$. Each column has +1 at its tail and -1 at its head.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} x_{12} \\ x_{13} \\ x_{23} \\ x_{24} \\ x_{34} \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 0 \\ 0 \\ -10 \end{pmatrix}.$$

> This A is the **node–arc incidence matrix**. Multiplying a column by its flow subtracts that amount at the receiving node and adds it at the sending node in the outgoing-minus-incoming convention.

Every column sums to zero, so the four balance rows are dependent.

One balance equation is redundant

Adding all four balances gives 0 = 10 + 0 + 0 - 10. If the first three hold, the fourth holds automatically.

> For a network with n nodes, connected when arrow directions are ignored, the incidence matrix has rank n - 1. Delete one node row to obtain n - 1 independent equalities.

Here delete node 4’s row. A simplex basis contains three independent arc columns.

From now on, matrix B uses this reduced three-row matrix. The graph and node balances still include node 4.

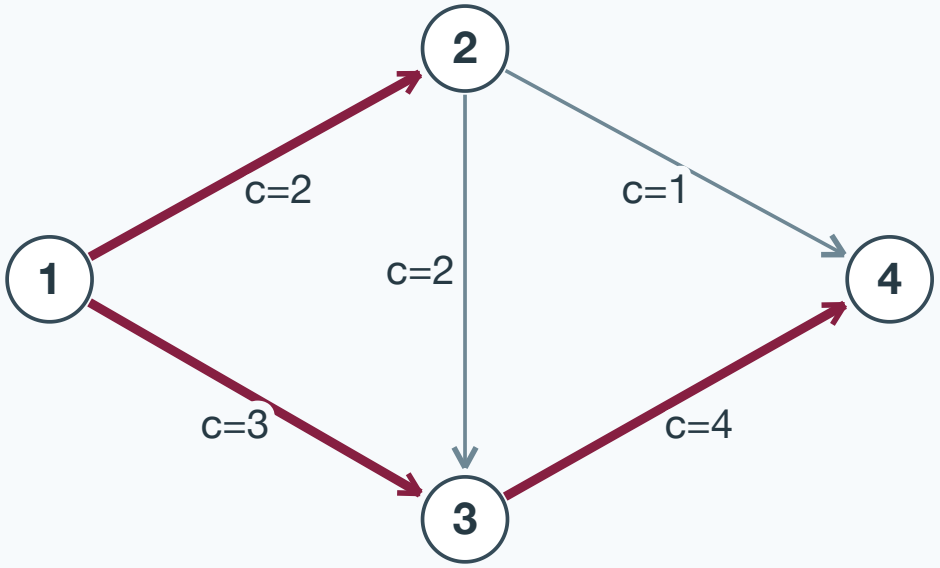
Appendix: why connectedness gives rank n - 1.

A spanning tree supplies exactly the basic columns

A **spanning tree** connects all nodes and contains no cycle when arrow directions are ignored. It has n - 1 arcs.

> Tree-basis theorem: n - 1 arc columns of the reduced incidence matrix are independent exactly when their arcs form a spanning tree.

Choose T = {12, 13, 34}. Set the other flows x_{23} and x_{24} to zero. Then solve the balance equations for the tree flows.



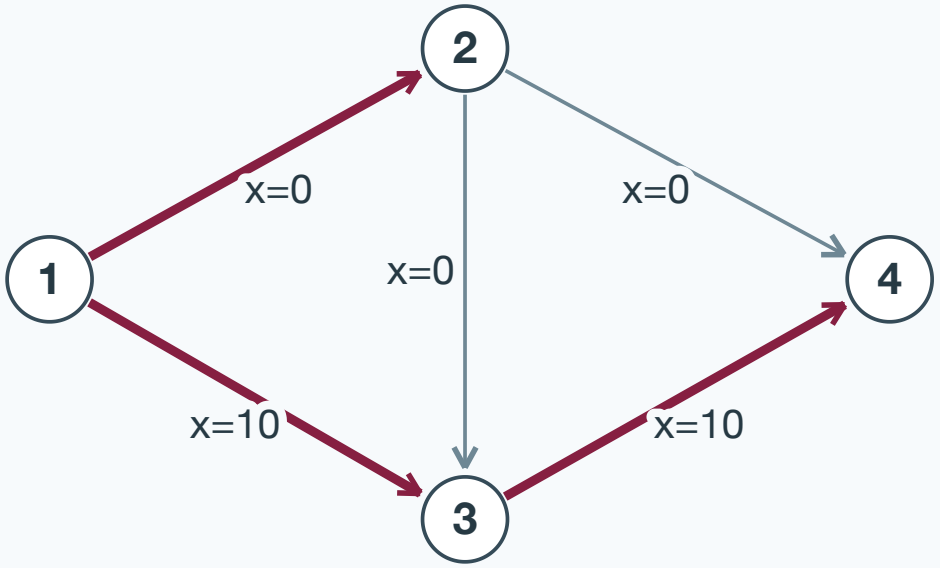
Thick maroon arcs: chosen basic columns. Direction still determines signs in the balance equations.

Solve the initial tree flows at its leaves

The nonbasic flows x_{23} and x_{24} are zero. Use the balances at nodes 2, 1, and 3, in that order.

$$\begin{aligned} -x_{12} &= 0 && \Rightarrow x_{12} = 0, \\ x_{12} + x_{13} &= 10 && \Rightarrow x_{13} = 10, \\ -x_{13} + x_{34} &= 0 && \Rightarrow x_{34} = 10. \end{aligned}$$

All basic flows are nonnegative. The basic flow x_{12} is zero, so this basic feasible solution is degenerate.



Arc labels are current flows. Thick maroon arcs form the basis tree; thin arcs are nonbasic.

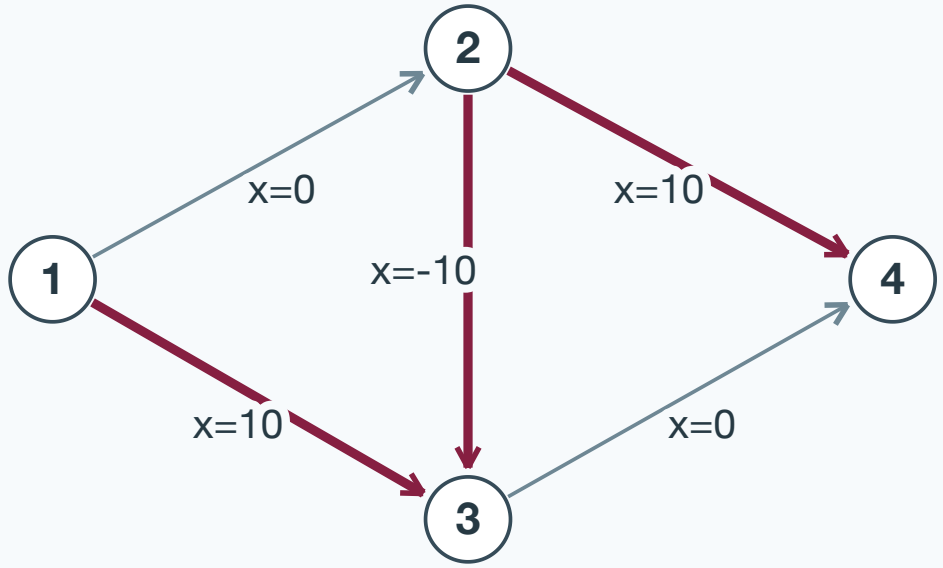
The same network, but an infeasible tree basis

Ship 10 units from node 1 to node 4.

Choose $T = \{13, 23, 24\}$. Set the nonbasic flows $x_{12} = x_{34} = 0$.

$$\begin{aligned} \text{Node } 1: &\quad x_{13} = 10, \\ \text{Node } 4: &\quad -x_{24} = -10 \;\Rightarrow x_{24} = 10, \\ \text{Node } 2: &\quad x_{23} + x_{24} = 0 \;\Rightarrow x_{23} = -10. \end{aligned}$$

The negative x_{23} would send 10 units against the arrow $2 \to 3$. It violates $x_{23} \geq 0$.



Maroon: basic arcs. Labels: values obtained from the balance equations.

This tree is infeasible; the problem is feasible. The previous tree $\{12, 13, 34\}$ gives $x_{13} = x_{34} = 10$, all other flows zero. We return to that feasible tree next.

The full initial dictionary comes from substitution

Send 10 units from node 1 to node 4. Minimize $f = 2x_{12} + 3x_{13} + 2x_{23} + x_{24} + 4x_{34}$; all flows are nonnegative.

Return to the feasible tree {12, 13, 34}. Keep its nonbasic flows x_{23} and x_{24} symbolic. Node 2 gives $x_{12} = x_{23} + x_{24}$; substitute into node 1, then use node 4. Substitute the resulting flows into the original objective.

$$\begin{aligned} x_{12} &= x_{23} + x_{24}, \\ x_{13} &= 10 - x_{23} - x_{24}, \\ x_{34} &= 10 - x_{24}, \\ f &= 70 + x_{23} - 4x_{24}. \end{aligned}$$

The constant is $3(10) + 4(10) = 70$. The coefficients are $2 - 3 + 2 = 1$ for x_{23} and $2 - 3 + 1 - 4 = -4$ for x_{24}.

The matrix computation gives exactly the same dictionary

Delete node 4's row: the remaining right-hand side is $b_{\mathrm{red}} = (10, 0, 0)^{\top}$. Order basic variables (x_{12}, x_{13}, x_{34}) and nonbasic variables (x_{23}, x_{24}).

$$B = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad A_N = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$B^{-1} b_{\mathrm{red}} = \begin{pmatrix} 0 \\ 10 \\ 10 \end{pmatrix}, \quad B^{-1} A_N = \begin{pmatrix} -1 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$x_B = B^{-1} b_{\mathrm{red}} - B^{-1} A_N x_N.$$

The two products give the same three basic-variable equations as substitution.

Write the dual of our shipping problem

Restore node 4's redundant balance. Let p_i be the dual variable for node i.

Primal: same shipping problem

$$
\begin{aligned}
\min \quad & 2x_{12} + 3x_{13} + 2x_{23} \\
& \quad + x_{24} + 4x_{34} \\
& x_{12} + x_{13} = 10 && [p_1] \\
& -x_{12} + x_{23} + x_{24} = 0 && [p_2] \\
& -x_{13} - x_{23} + x_{34} = 0 && [p_3] \\
& -x_{24} - x_{34} = -10 && [p_4] \\
& x_{12}, x_{13}, x_{23}, x_{24}, \\
& \quad x_{34} \geq 0.
\end{aligned}
$$

Dual: one constraint per flow

$$
\begin{aligned}
\max \quad & 10p_1 - 10p_4 \\
& p_1 - p_2 \leq 2 && [x_{12}] \\
& p_1 - p_3 \leq 3 && [x_{13}] \\
& p_2 - p_3 \leq 2 && [x_{23}] \\
& p_2 - p_4 \leq 1 && [x_{24}] \\
& p_3 - p_4 \leq 4 && [x_{34}] \\
& p_i \in \mathbb{R} \quad (i = 1, \dots, 4).
\end{aligned}
$$

For x_{12}: coefficients +1 at node 1 and –1 at node 2, with unit cost 2, give $p_1 - p_2 \leq 2$.

Equality balances give unrestricted dual variables. Nonnegative primal flows give ≤ dual constraints.

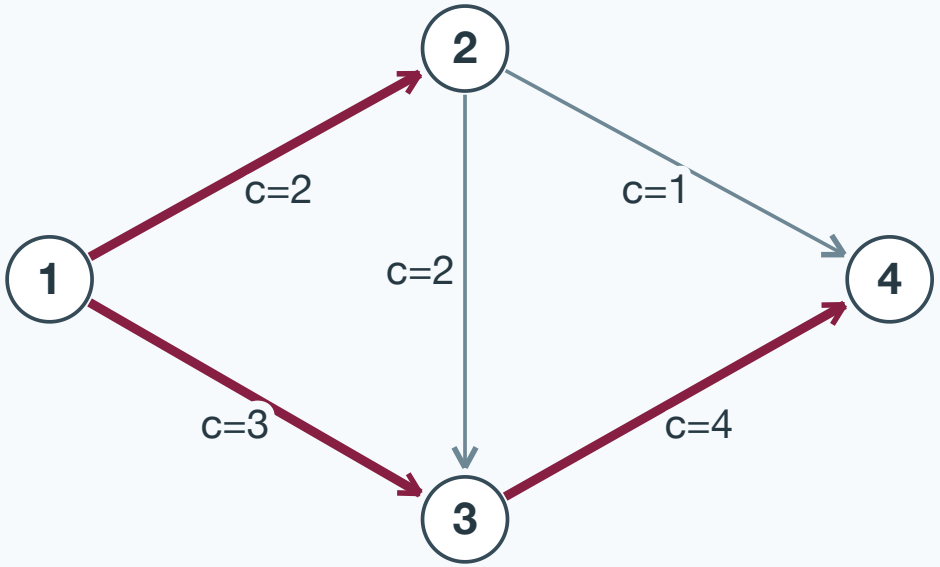
Compute a dual candidate from the current tree

Use the feasible tree $T = \{12, 13, 34\}$. As in ordinary simplex, choose p so the basic reduced costs are zero: $p_i - p_j = c_{ij}$ on each tree arc.

Choose $p_1 = 0$. Adding a constant to all four p_i changes neither their differences nor $10p_1 - 10p_4$.

$$\begin{aligned} p_1 - p_2 = 2 \qquad &\Rightarrow p_2 = -2, \\ p_1 - p_3 = 3 \qquad &\Rightarrow p_3 = -3, \\ p_3 - p_4 = 4 \qquad &\Rightarrow p_4 = -7. \end{aligned}$$

The candidate is $p = (0, -2, -3, -7)$. Node dual variables are also called **node potentials**.



Arc labels: unit costs. Maroon: basic arcs.

This is a dual candidate. Check the two nonbasic-arc inequalities next.

Check the dual inequalities and reduced costs

Send 10 units from node 1 to node 4. Minimize $f = 2x_{12} + 3x_{13} + 2x_{23} + x_{24} + 4x_{34}$; all flows are nonnegative.

Candidate $p = (0, -2, -3, -7)$, in node order 1–4. Only the two nonbasic-arc inequalities remain:

Arc	Dual constraint check	Cost minus dual expression
2 → 3	$p_2 - p_3 = -2 - (-3) = 1 \le 2$	$2 - 1 = 1$
2 → 4	$p_2 - p_4 = -2 - (-7) = 5 > 1$	$1 - 5 = -4$

In general, an arc's dual constraint is $p_i - p_j \le c_{ij}$. Its **reduced cost** is $\bar{c}_{ij} = c_{ij} - (p_i - p_j)$.

$$f = 70 + \underbrace{1}_{\bar{c}_{23}}\, x_{23}\, \underbrace{-4}_{\bar{c}_{24}}\, x_{24}.$$

Arc 24 fails the dual feasibility check. The reduced costs match the dictionary coefficients. Choose x_{24} to enter; next find its feasible step.

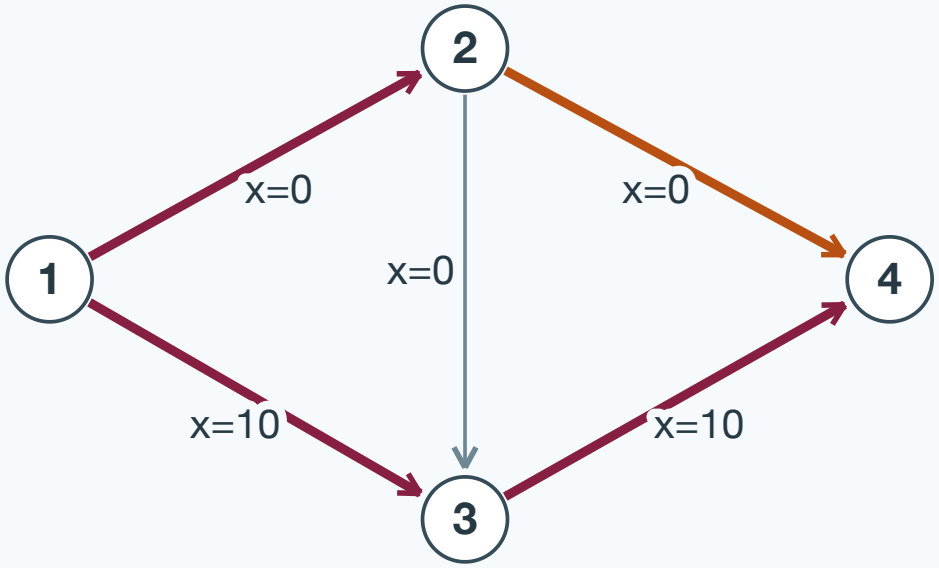
Adding arc 2 → 4 creates exactly one cycle

Current tree: $T = \{12, 13, 34\}$. Add only arc $2 \to 4$; keep the other nonbasic flow $x_{23} = 0$.

Ignoring arrow directions, the tree has exactly one path from 2 to 4: **2 — 1 — 3 — 4**.

> Two different paths would already form a cycle, contradicting the definition of a tree.

Adding $2 \to 4$ closes that path into one cycle: **2 — 4 — 3 — 1 — 2**.



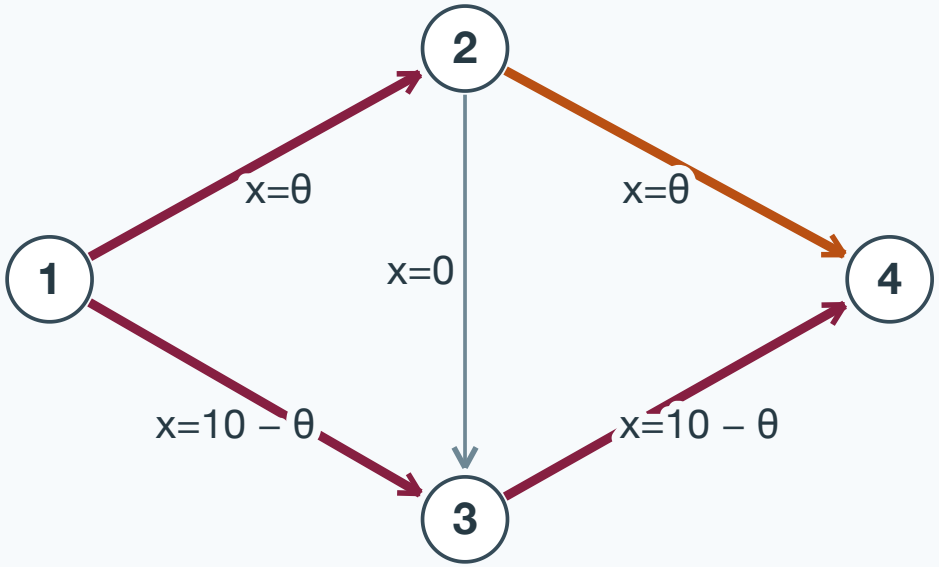
Maroon: tree arcs. Orange: entering arc. The thin gray arc 23 stays at zero.

This cycle describes connections, not a directed shipping route. Every original arrow remains unchanged.

Reroute flow while preserving every node balance

Increase x_{24} from 0 to $\theta \geq 0$, holding $x_{23} = 0$. Reroute θ units from $1 \to 3 \to 4$ to $1 \to 2 \to 4$.

Arc	New flow	Balance reason
$2 \to 4$	θ	Chosen entering flow
$1 \to 2$	θ	Node 2 needs θ incoming.
$1 \to 3$	$10 - \theta$	Node 1 still sends 10.
$3 \to 4$	$10 - \theta$	Node 3 sends what it receives.
$2 \to 3$	0	Other nonbasic flow stays fixed.



The arrows are original arcs. A decrease sends less flow along an arc; it does not reverse the flow.

Node 4 still receives $\theta + (10 - \theta) = 10$. Next: find the largest θ that keeps every flow nonnegative.

Your turn: how far can we send flow around the cycle?

$$
\begin{aligned}
x_{12} &= x_{23} + x_{24}, \\
x_{13} &= 10 - x_{23} - x_{24}, \\
x_{34} &= 10 - x_{24}, \\
f &= 70 + x_{23} - 4x_{24}.
\end{aligned}
$$

Work for about 5 minutes. Hold $x_{23} = 0$ and set $x_{24} = \theta \geq 0$.

1. Which basic flows decrease?
2. What is the largest feasible θ?
3. Which arc can leave? Compute the new total cost.

Use the full flow-change table, not only the entering arc.

Both decreasing flows reach zero at θ = 10

$$10 - \theta \geq 0 \quad \Rightarrow \quad \theta^* = 10.$$

Arcs 13 and 34 reach zero together. Break the tie by choosing arc 13 to leave.

New flows, in arc order (12, 13, 23, 24, 34): $x = (10, 0, 0, 10, 0)$.

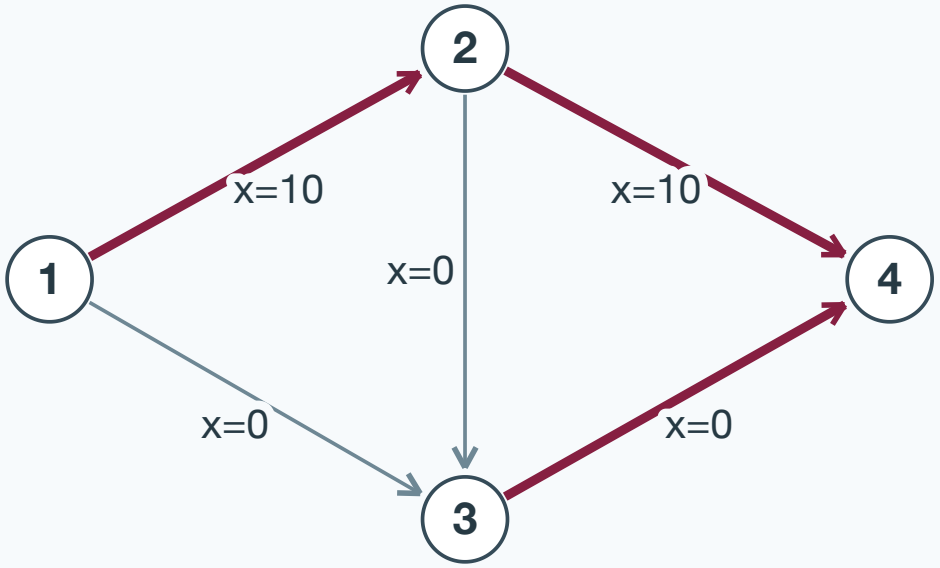
Cycle cost per unit: $1 + 2 - 4 - 3 = -4$, the entering reduced cost.

$$f_{\text{new}} = 70 - 4(10) = 30.$$

The new tree is {12, 24, 34}, with arc 34 basic at zero. Both bases are degenerate, yet this pivot made a positive move.

Check the new flows and write the entire dictionary

$$
\begin{aligned}
x_{12} &= 10 - x_{13}, \\
x_{24} &= 10 - x_{13} - x_{23}, \\
x_{34} &= x_{13} + x_{23}, \\
f &= 30 + 4x_{13} + 5x_{23}.
\end{aligned}
$$



Arc labels are current flows. Thick maroon arcs form the basis tree; thin arcs are nonbasic.

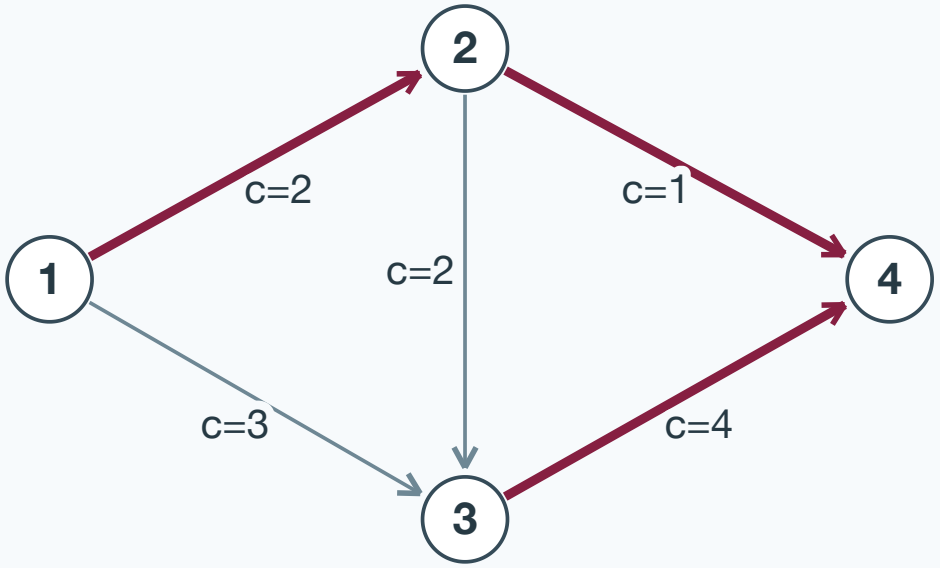
Solve the old x_{13} equation for x_{24}: x_{24} = 10 - x_{13} - x_{23}. Substitute it into the other rows and the objective. Both nonbasic objective coefficients are nonnegative.

The new tree gives a feasible dual

Tree T = {12, 24, 34}; set $p_1 = 0$.

$$
\begin{aligned}
p_1 - p_2 &= 2 && \Rightarrow p_2 = -2,\\
p_2 - p_4 &= 1 && \Rightarrow p_4 = -3,\\
p_3 - p_4 &= 4 && \Rightarrow p_3 = 1.
\end{aligned}
$$

$$
\begin{aligned}
\bar{c}_{13} &= 3 - 0 + 1 = 4,\\
\bar{c}_{23} &= 2 - (-2) + 1 = 5.
\end{aligned}
$$



Arrow: original arc direction. Labels: cost per unit. Node balances are stated beside the graph.

Tree reduced costs are zero; the two other reduced costs are positive. Every dual inequality now holds.

The final certificate is an arithmetic equality

Send 10 units from node 1 to node 4. Minimize $f = 2x_{12} + 3x_{13} + 2x_{23} + x_{24} + 4x_{34}$; all flows are nonnegative.

Flows: arc order (12, 13, 23, 24, 34)	Dual variables: node order (1, 2, 3, 4)
$x = (10, 0, 0, 10, 0)$	$p = (0, -2, 1, -3)$

$$c^\top x = 2(10) + 1(10) = 30, \qquad b^\top p = 10(0) - 10(-3) = 30.$$

No feasible flow can cost less than the feasible dual value 30. The displayed flow attains 30, so it is optimal. This completes the shipping example.

Network simplex: the familiar five steps

Step	Uncapacitated minimum-cost flow
1. Current basis	A spanning tree with nonnegative flows
2. Dual calculation	Set one p value; solve $p_i - p_j = c_{ij}$ on tree arcs
3. Entering arc	A nonbasic arc with reduced cost $c_{ij} - p_i + p_j < 0$
4. Leaving arc	Follow its cycle until a decreasing flow reaches zero; if none decrease, the LP is unbounded
5. Update	Exchange arcs, update flows, and recompute p

> If all reduced costs are nonnegative, stop: the current basic feasible solution is optimal.

What if no cycle arc needs to decrease?

If every cycle arrow agrees with the traversal, all cycle flows increase. With no upper capacities, no nonnegativity constraint limits θ.

$$f(\theta) = f(0) + \theta \bar{c}_{\text{enter}}, \quad \bar{c}_{\text{enter}} < 0, \quad \theta \geq 0.$$

> The cost tends to -∞: the minimum-cost flow LP is unbounded below. The improving directed cycle is a certificate.

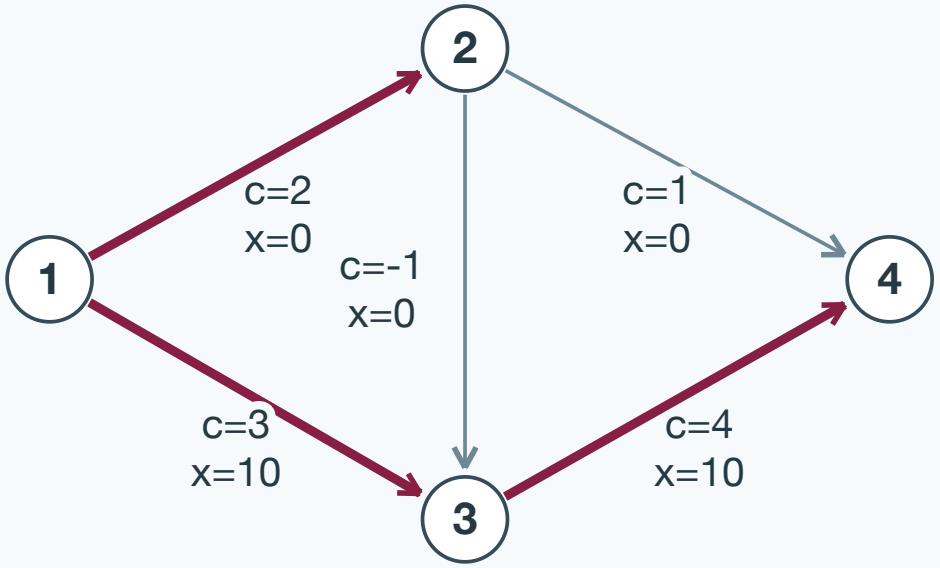
Example: b = 0, arcs 1 → 2 of cost -2, 2 → 3 of cost 0, and 3 → 1 of cost 1. Sending θ around this cycle costs -θ and changes no node balance.

Practice: choose the right arc to enter

Return to the initial tree {12, 13, 34}, but change only the cost of arc 23 from 2 to -1. In arc order (12, 13, 23, 24, 34), the costs are (2, 3, -1, 1, 4).

In that arc order, initial flows remain (0, 10, 0, 0, 10), cost 70. Basic costs did not change, so p = (0, -2, -3, -7), in node order 1–4.

> Work for about 5 minutes: compute both nonbasic reduced costs. Use the most-negative rule, perform one pivot, and decide whether to stop.



On each arc: c = cost per unit; x = initial flow. Thick maroon arcs form the initial basis tree.

Practice solution

The new reduced costs are $\bar{c}_{23} = -1 - (-2) - 3 = -2$ and $\bar{c}_{24} = -4$. The most-negative rule still chooses arc 24.

The cycle and ratio test are unchanged: θ = 10, arc 13 leaves, flows become (10, 0, 0, 10, 0).

$$p = (0, -2, 1, -3), \qquad \bar{c}_{13} = 4, \quad \bar{c}_{23} = -1 + 2 + 1 = 2.$$

Both nonbasic reduced costs are nonnegative; the cost is still 30 and the same dual value proves optimality.

Why use the graph instead of a dense inverse?

Calculation	Tree-based work
Basic flows	Remove leaves and solve one balance at a time
Basis dual variables	Traverse tree; one new p value per arc
Entering arc	Scan nonbasic reduced costs
Direction and ratio	Find the unique cycle and inspect its arcs

A simple implementation uses O(n + m) work per iteration: n nodes, m arcs. Specialized tree data structures can improve update costs.

> This is a bound on one iteration. It does not give a polynomial bound on the total number of simplex pivots.

A graph pivot and a dictionary pivot are the same operation

The tree identifies B. The cycle gives a direction satisfying Ad = 0. The ratio test preserves nonnegativity. Reduced costs predict the objective change.

Next: arc capacities can stop an increasing flow. We will also study zero-length pivots and how to obtain a starting feasible tree.

Appendix: complete structural proofs and an additional two-pivot example.

Appendix: proofs and additional calculations

Why tree columns form a basis

Why cycle directions preserve balances

Why reduced cost equals cycle cost

Primal–dual optimality identity

A complete example with negative arc costs

Tree columns are independent; cycle columns are dependent

On a tree, suppose a linear combination of incidence columns is zero. A leaf row involves exactly one tree column, whose coefficient must be zero. Remove that leaf and repeat. All coefficients are zero.

This remains true after one row is removed: the missing row is the negative sum of the others. Thus n - 1 tree columns form an invertible square reduced matrix.

Conversely, columns on any undirected cycle have a nonzero signed combination equal to zero. Follow the cycle, using +1 for agreeing arrows and -1 for opposing arrows.

> A set of n - 1 arcs with no cycle is a spanning tree. A connected graph has such a tree, so its incidence rank is n - 1.

A cycle direction solves the same basis equations

Give the entering arc direction +1. Each cycle node receives and sends the same change; all other nodes are unchanged. Therefore Ad = 0.

$$Bd_B + A_{\text{enter}} = 0 \quad \Rightarrow \quad d_B = -B^{-1}A_{\text{enter}}.$$

Because B is invertible, the graph direction and the algebraic direction are identical. The cycle entries are +1 or -1, and arcs outside the cycle have direction 0.

> For uncapacitated flows, θ is limited only by $d_e = -1$. If no such arc exists, every θ ≥ 0 remains feasible.

The cycle cost is the entering reduced cost

Let A_T contain the tree columns and c_T their costs. The basis equalities are $A_T^\top p = c_T$, so $c_e = p^\top A_e$ on each tree arc. For a cycle direction d with entering component 1 and $Ad = 0$,

$$\begin{aligned} c^\top d &= c_{\text{enter}} + \sum_{e \in T} c_e d_e \\ &= c_{\text{enter}} + p^\top \sum_{e \in T} A_e d_e \\ &= c_{\text{enter}} - p^\top A_{\text{enter}} = \bar{c}_{\text{enter}}. \end{aligned}$$

Hence f(x + θd) = f(x) + θ times the entering reduced cost. The signed sum of cycle costs and the dictionary objective coefficient agree.

Nonnegative reduced costs prove optimality

For any feasible flow y, $Ay = b$. With $\bar{c} = c - A^\top p$,

$$c^\top y = p^\top A y + \bar{c}^\top y = p^\top b + \bar{c}^\top y.$$

If all reduced costs are nonnegative and y ≥ 0, every feasible objective is at least $p^\top b$. The current tree solution has zero nonbasic flows and zero basic reduced costs, so it attains $p^\top b$.

> This uses basis equalities and feasibility. A zero basic flow creates no exception to the proof.

Additional example: negative costs, two pivots

Four nodes have balances b = (4, 2, -1, -5), in node order 1–4. All flows are nonnegative with no upper bounds.

Arc order	12	13	14	23	24	32	34
Cost	2	-5	7	6	4	-1	3

Initial tree: {12, 23, 24}. The balances give $x_{12} = 4$, $x_{23} = 1$, $x_{24} = 5$; all other flows are zero. For this example, fix $p_4 = 0$ and solve the tree equalities.

$$f = 2(4) + 6(1) + 4(5) = 34, \qquad p = (6, 4, -2, 0).$$

The nonbasic reduced costs in order (13, 14, 32, 34) are (-13, 1, 5, 5). Enter 13.

Pivot 1: enter 13, leave 23

Follow 1 → 3 → 2 → 1. Increase x_{13}; decrease x_{23} and x_{12}.

$$\theta = \min\{x_{23}, x_{12}\} = \min\{1, 4\} = 1.$$

Flow order	12	13	14	23	24	32	34
After pivot 1	3	1	0	0	5	0	0

$$f = 34 - 13 = 21, \qquad p = (6, 4, 11, 0).$$

Tree {12, 13, 24}. Nonbasic reduced costs (14, 23, 32, 34) = (1, 13, -8, -8). Break the tie by choosing arc 32.

Pivot 2: enter 32, leave 12

Follow 3 → 2 → 1 → 3. Increase x_{32} and x_{13}; decrease x_{12}. The step is θ = 3.

Flow order	12	13	14	23	24	32	34
Final flow	0	4	0	0	5	3	0

$$f = 21 - 8(3) = -3, \qquad p = (-2, 4, 3, 0).$$

Tree {13, 24, 32}. Nonbasic reduced costs (12, 14, 23, 34) = (8, 9, 5, 0), all nonnegative.

$$b^{\top} p = 4(-2) + 2(4) - 1(3) - 5(0) = -3.$$

The final flow is optimal. The zero reduced cost of arc 34 permits an alternative optimal flow if its cycle admits a positive step.