Column Generation and Constraint Generation

A linear program can have too many variables or constraints to include them all at once.

> **Keep a smaller LP. Use its solution to decide what to add.**

We will complete a cutting-stock example, then add violated constraints to a second LP. Pause at the practice prompts; answers follow on the next page.

- Column generation: cutting stock
- Constraint generation: a geometric example
- Proofs and optional decomposition connections

Can we solve an LP without listing everything?

So far, we have written down all variables and constraints before solving an LP. Some applications have too many to list conveniently.

Many possible variables: start with a few, solve the smaller LP, and use its dual solution to identify another variable worth adding. This is **column generation**.

Many constraints: start with some, solve the smaller LP, and check whether its solution violates an omitted constraint. Add such a constraint and solve again. This is **constraint generation**.

Our goal: learn what to add—and how to prove that the final answer is optimal for the complete LP.

Example: a paper mill must fill customer orders

A paper mill has rolls of width 10. Cutting a roll lengthwise produces narrower rolls of the same length; we call these pieces. All widths use the same unit.

Width of each piece	Pieces ordered
3	4
4	3
6	2

Every original roll has the same cost. Unused width is discarded.

Original roll: width 10



One possible cut



Three pieces; gray is unused width.

Widths add up to 3 + 3 + 3 + 1 = 10.

Decision: how should we cut the rolls, and how many rolls should we use? Meet every order using as few original rolls as possible. Extra pieces are allowed.

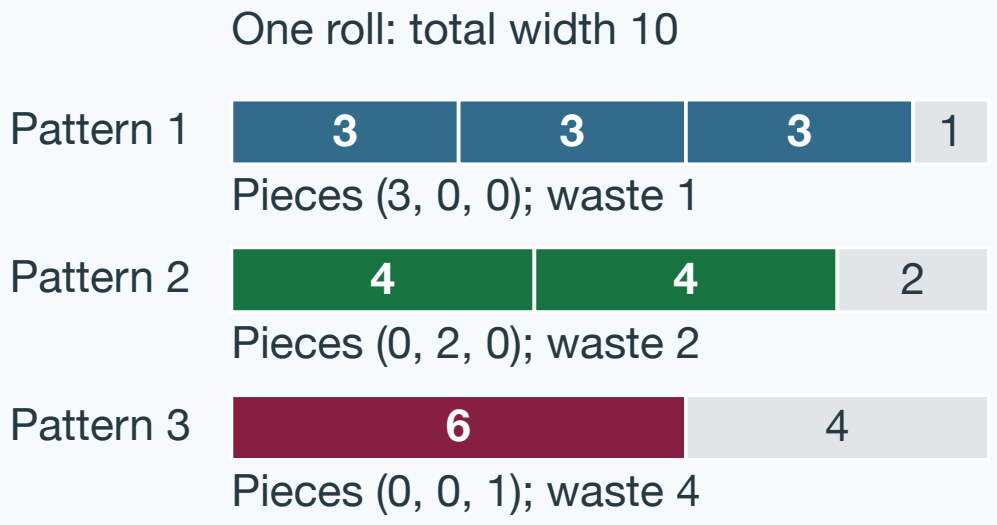
One cutting pattern becomes one LP variable

A **cutting pattern** describes the entire cut of one roll. Start with these three choices; other patterns can be added later.

Pattern j has fixed piece counts $a^{(j)}$, in width order $(3, 4, 6)$. For example, $a^{(1)} = (3, 0, 0)$.

> x_j counts the rolls using pattern j. Pattern 1 therefore produces $3x_1$ width-3 pieces.

Each pattern supplies one column of LP coefficients. A feasible pattern a satisfies $3a_1 + 4a_2 + 6a_3 \leq 10$, with nonnegative integer piece counts.



Numbers inside pieces are widths. Gray is unused width.

Write the LP using only these three patterns

Patterns 1, 2, 3 produce (3,0,0), (0,2,0), (0,0,1) pieces per roll, respectively. Orders are 4 width-3, 3 width-4, and 2 width-6 pieces.

$$
\begin{aligned}
\min \quad & x_1 + x_2 + x_3 \\
\text{s.t.} \quad & 3x_1 \geq 4, \\
& 2x_2 \geq 3, \\
& x_3 \geq 2, \\
& x_1, \dots, x_3 \geq 0.
\end{aligned}
$$

The row $3x_1 \geq 4$ counts width-3 pieces. Each pattern already fits width 10; here we choose how many rolls use it.

Restricted master problem: keep all demand constraints, but start with only these three patterns.

The **continuous LP relaxation** allows fractional x_j. For the same included patterns, this cannot increase the minimum.

The first solution uses 29/6 rolls in the LP

$$x_1 = \frac{4}{3}, \qquad x_2 = \frac{3}{2}, \qquad x_3 = 2, \qquad z = \frac{4}{3} + \frac{3}{2} + 2 = \frac{29}{6}.$$

Produced pieces	Calculation	Demand
Width 3	$3(4/3) = 4$	4
Width 4	$2(3/2) = 3$	3
Width 6	$1(2) = 2$	2

These pattern quantities are feasible for the full pattern LP: set every omitted pattern quantity to zero. More available patterns may lower its cost.

The three-pattern LP and its dual

Demand: 4, 3, 2 pieces of widths 3, 4, 6; dual variables: p_1, p_2, p_3, respectively.

Primal · patterns 1–3

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 \\ \text{s.t.} \quad & 3x_1 \geq 4, \\ & 2x_2 \geq 3, \\ & x_3 \geq 2, \\ & x_1, \dots, x_3 \geq 0. \end{aligned}$$

Dual · one inequality per pattern

$$\begin{aligned} \max \quad & 4p_1 + 3p_2 + 2p_3 \\ \text{s.t.} \quad & 3p_1 \leq 1, \\ & 2p_2 \leq 1, \\ & p_3 \leq 1, \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

Pattern 1 gives $3p_1 \leq 1$; patterns 2 and 3 give $2p_2 \leq 1$ and $p_3 \leq 1$. Each right-hand side is 1 because one use of a pattern uses one roll.

$$x = (4/3, 3/2, 2), \qquad p = (1/3, 1/2, 1), \qquad \text{both values} = 29/6.$$

Optimal with these three patterns. Could another cutting pattern reduce the number of rolls?

Another way to cut one roll

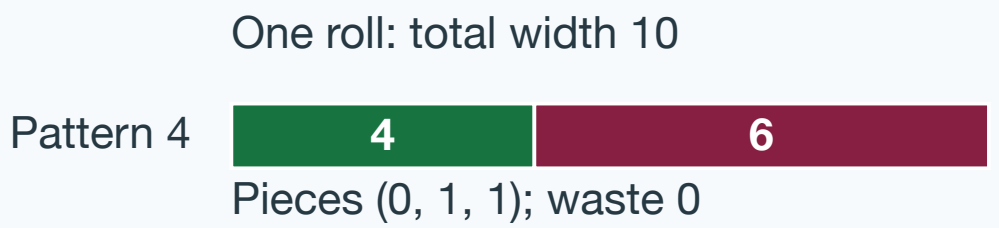
The first three patterns are starting choices. A roll may also contain pieces of different widths.

$$4 + 6 = 10, \qquad a^{(4)} = (0, 1, 1).$$

Pattern 4 gives no width-3 pieces, one width-4 piece, and one width-6 piece.

> Let x_4 be the number of rolls using pattern 4. It adds x_4 to each of the last two demand rows and to the total rolls used.

Keep patterns 1–3 as available choices. Next, write the four-pattern LP and its dual.



Numbers inside pieces are widths. Gray is unused width.

Adding a pattern adds a dual constraint

Orders: $(4, 3, 2)$ in width order $(3, 4, 6)$. The new variable x_4 is bold.

Primal · patterns 1–4

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + \boldsymbol{x_4} \\ \text{s.t.} \quad & 3x_1 \geq 4, \\ & 2x_2 + \boldsymbol{x_4} \geq 3, \\ & x_3 + \boldsymbol{x_4} \geq 2, \\ & x_1, \dots, \boldsymbol{x_4} \geq 0. \end{aligned}$$

Dual · one inequality per pattern

$$\begin{aligned} \max \quad & 4p_1 + 3p_2 + 2p_3 \\ \text{s.t.} \quad & 3p_1 \leq 1, \\ & 2p_2 \leq 1, \\ & p_3 \leq 1, \\ & \boxed{p_2 + p_3 \leq 1}, \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

Column of x_4: $(0, 1, 1)^\top$; objective coefficient: 1. Hence $0p_1 + p_2 + p_3 \leq 1$.

The left-hand side is $p^\top a^{(4)} = 0p_1 + p_2 + p_3$: multiply each column entry by its demand dual variable and add.

Does the previous dual solution still work?

From the three-pattern LP: $p^{\text{old}} = (1/3, 1/2, 1)$. Test it in the four-pattern dual below.

Primal · patterns 1–4

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 \\ \text{s.t.} \quad & 3x_1 \geq 4, \\ & 2x_2 + x_4 \geq 3, \\ & x_3 + x_4 \geq 2, \\ & x_1, \dots, x_4 \geq 0. \end{aligned}$$

Dual · one inequality per pattern

$$\begin{aligned} \max \quad & 4p_1 + 3p_2 + 2p_3 \\ \text{s.t.} \quad & 3p_1 \leq 1, \\ & 2p_2 \leq 1, \\ & p_3 \leq 1, \\ & p_2 + p_3 \leq 1, \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

$$(p^{\text{old}})^\top a^{(4)} = 0\left(\frac{1}{3}\right) + 1\left(\frac{1}{2}\right) + 1(1) = \frac{3}{2} > 1.$$

The old dual solution fails the new inequality. Pattern 4 itself still fits the roll.

Reduced cost: $\bar{c}_4 = 1 - (p^{\text{old}})^\top a^{(4)} = 1 - \frac{3}{2} = -\frac{1}{2}$. Solve again with pattern 4.

With pattern 4, the optimum decreases to 23/6

Orders: 4, 3, 2 pieces of widths 3, 4, 6. Pattern 4 gives one width-4 and one width-6 piece.

Primal · patterns 1–4

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 \\ \text{s.t.} \quad & 3x_1 \geq 4, \\ & 2x_2 + x_4 \geq 3, \\ & x_3 + x_4 \geq 2, \\ & x_1, \dots, x_4 \geq 0. \end{aligned}$$

Dual · one inequality per pattern

$$\begin{aligned} \max \quad & 4p_1 + 3p_2 + 2p_3 \\ \text{s.t.} \quad & 3p_1 \leq 1, \\ & 2p_2 \leq 1, \\ & p_3 \leq 1, \\ & p_2 + p_3 \leq 1, \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

$$x = (4/3, 1/2, 0, 2), \qquad p = (1/3, 1/2, 1/2), \qquad \text{both values} = 23/6.$$

Demand checks: $3(4/3) = 4$, $2(1/2) + 2 = 3$, $0 + 2 = 2$. All four dual inequalities also hold.

Optimal for patterns 1–4. Use the new dual variables to search for another pattern.

How do we find the next pattern?

Use the updated dual solution p. Each new pattern has cost 1, so its reduced cost is

$$\bar{c}(a) = 1 - p^\top a = 1 - (p_1 a_1 + p_2 a_2 + p_3 a_3).$$

> Search for the largest $p^\top a$ among patterns satisfying $3a_1 + 4a_2 + 6a_3 \le 10$, with nonnegative integer piece counts. This search is called **pricing**.

If $p^\top a > 1$, the pattern has negative reduced cost: add it and solve again. Here the patterns are small enough to list.

Negative reduced cost identifies a candidate. Degeneracy can still prevent an immediate strict improvement.

Your turn: price the next pattern

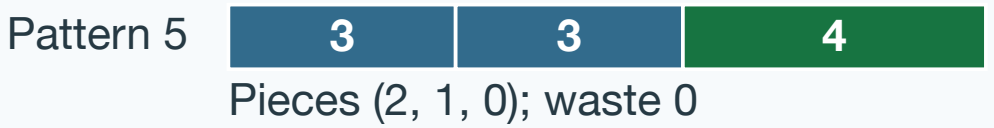
The current dual vector is $p = (1/3, 1/2, 1/2)$.

Consider pattern 5, $a^{(5)} = (2, 1, 0)$.

1. Check that the pieces fit within width 10.
2. Compute $p^\top a^{(5)}$.
3. Compute $1 - p^\top a^{(5)}$.
4. Should this column be added?

Pause for 5 minutes and write your four answers before checking the next page.

One roll: total width 10



Pieces (2, 1, 0); waste 0

Numbers inside pieces are widths. Gray is unused width.

Pattern 5 has reduced cost −1/6

$$3(2) + 4(1) + 6(0) = 10.$$

$$p^\top a^{(5)} = 2\left(\frac{1}{3}\right) + \frac{1}{2} = \frac{7}{6}, \qquad \bar{c}(a^{(5)}) = 1 - \frac{7}{6} = -\frac{1}{6}.$$

Add pattern 5. For these dual variables, it is the pattern with the largest value in the complete feasible-pattern list.

The full list and its coverage are shown after the next reoptimization.

The restricted LP now has five pattern variables

$$
\begin{aligned}
\min \quad & x_1 + x_2 + x_3 + x_4 + x_5 \\
\text{s.t.} \quad & 3x_1 + 2x_5 \geq 4, \\
& 2x_2 + x_4 + x_5 \geq 3, \\
& x_3 + x_4 \geq 2, \qquad x \geq 0.
\end{aligned}
$$

A new optimal solution is $x = (2/3, 0, 0, 2, 1)$, with $z = 11/3$.

Check production: $3(2/3) + 2 = 4$, $2 + 1 = 3$, and $2 = 2$.

Find the dual solution for the five-pattern LP

Demand is $(4, 3, 2)$ in width order $(3, 4, 6)$. The solution $x = (2/3, 0, 0, 2, 1)$ uses patterns 1, 4, and 5.

Primal · patterns 1–5

$$\begin{aligned} \min \quad & x_1 + x_2 + x_3 + x_4 + x_5 \\ \text{s.t.} \quad & 3x_1 + 2x_5 \geq 4, \\ & 2x_2 + x_4 + x_5 \geq 3, \\ & x_3 + x_4 \geq 2, \\ & x_1, \dots, x_5 \geq 0. \end{aligned}$$

Dual · one inequality per pattern

$$\begin{aligned} \max \quad & 4p_1 + 3p_2 + 2p_3 \\ \text{s.t.} \quad & 3p_1 \leq 1, \\ & 2p_2 \leq 1, \\ & p_3 \leq 1, \\ & p_2 + p_3 \leq 1, \\ & 2p_1 + p_2 \leq 1, \\ & p_1, p_2, p_3 \geq 0. \end{aligned}$$

Their quantities are positive, so complementary slackness makes their dual inequalities equalities:

$$3p_1 = 1, \quad p_2 + p_3 = 1, \quad 2p_1 + p_2 = 1 \quad \Longrightarrow \quad p = (1/3, 1/3, 2/3).$$

Both values are $11/3$; all five dual inequalities hold. Next: check every omitted pattern.

Check every pattern that could give a larger value

At $p = (1/3, 1/3, 2/3)$, the relevant pattern values are:

Pattern	Piece counts	Used width	$p^\top a$
1	(3,0,0)	9	1
2	(0,2,0)	8	$2/3$
3	(0,0,1)	6	$2/3$
4	(0,1,1)	10	1
5	(2,1,0)	10	1
6	(1,0,1)	9	1

> Every other feasible pattern can be filled out to one of these. Because the dual variables are nonnegative, adding a piece cannot decrease its dual value.

Thus no feasible pattern has value above 1. Pattern 6 has zero reduced cost and need not be added.

Matching bounds prove that we can stop

Our feasible pattern quantities use $11/3$ rolls. Therefore the minimum over *all* patterns is at most $11/3$.

For every feasible pattern, $p^\top a \le 1$. Weak duality gives a lower bound $4p_1 + 3p_2 + 2p_3 = 11/3$.

$$\boxed{\text{Full LP optimum} = \frac{11}{3}.}$$

We did not need to include every pattern. We did need to check that no omitted pattern invalidated the dual certificate.

Cutting-stock example: what we accomplished

We have finished the cutting-stock example: the minimum for the continuous LP, allowing every feasible pattern, is 11/3 rolls.

Patterns included	Minimum rolls in the LP	Next action
1–3	$29/6$	Add pattern 4: (0,1,1)
1–4	$23/6$	Add pattern 5: (2,1,0)
1–5	$11/3$	Stop: all pattern dual inequalities hold

What duality did: it identified patterns to add. At the end, the dual lower bound equaled the number of rolls in our feasible LP solution.

Next, write this same procedure for a general LP; then begin a separate example that adds constraints.

From cutting patterns to general LP variables

In our example, a_j contained pattern j's piece counts, $c_j = 1$ counted one roll, and $b = (4, 3, 2)^\top$ contained the orders. Let J be the indices of variables currently included.

Restricted primal · minimize

$$\begin{aligned} \min \quad & \sum_{j \in J} c_j x_j \\ \text{s.t.} \quad & \sum_{j \in J} a_j x_j \geq b, \\ & x_j \geq 0 \quad (j \in J). \end{aligned}$$

Its dual · maximize

$$\begin{aligned} \max \quad & b^\top p \\ \text{s.t.} \quad & a_j^\top p \leq c_j \quad (j \in J), \\ & p \geq 0. \end{aligned}$$

For an omitted variable, compute $\bar{c}_j = c_j - a_j^\top p$. Add a variable with $\bar{c}_j < 0$ and solve again. Stop when every omitted variable has $\bar{c}_j \geq 0$.

These signs apply to the displayed minimization primal with ≥ constraints and nonnegative variables.

New example: add constraints when the solution violates them

The cutting-stock example is complete. This is a separate LP with two variables. We will keep both variables but initially enforce only their bounds.

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5, \quad 2x_1 + x_2 \le 6, \\ & 0 \le x_1 \le 4, \quad 0 \le x_2 \le 4. \end{aligned}$$

> Removing constraints enlarges the feasible set. For this maximization problem, the smaller LP gives an **upper bound** on the complete LP optimum.

After each solve, check the omitted constraints. Add a violated constraint and solve again. The following plots show this same LP.

With only the box bounds, choose (4,4)

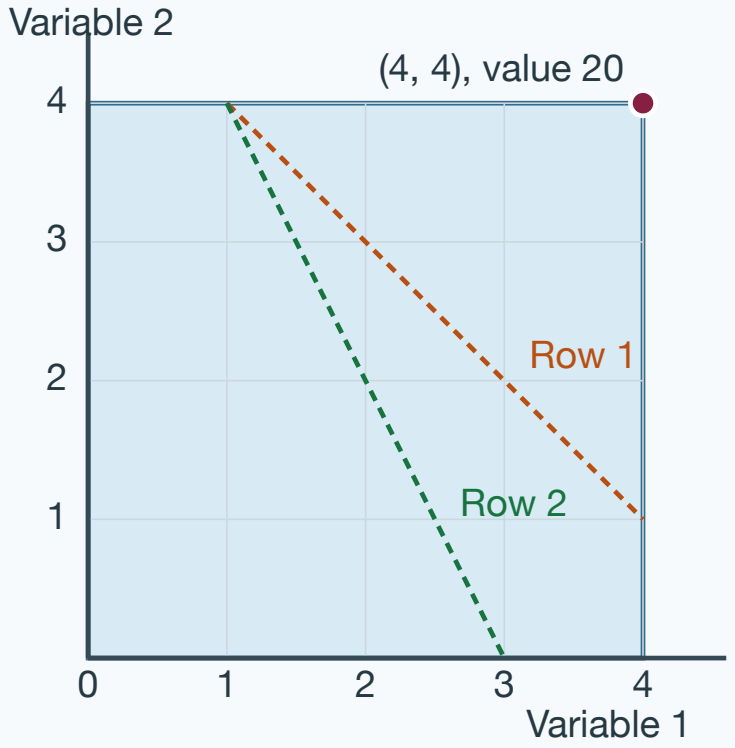
$$\max \; 3x_1 + 2x_2 \quad \text{s.t.} \quad 0 \le x_1, x_2 \le 4.$$

$$x = (4, 4), \qquad z = 20.$$

Check an omitted row:

$x_1 + x_2 = 8 > 5.$

The point is optimal for the box, but it is not feasible for the complete LP.



Horizontal: x_1; vertical: x_2. Orange: $x_1 + x_2 = 5$.
Green: $2x_1 + x_2 = 6$.

Add the violated row and solve again

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5, \qquad 0 \le x_1, x_2 \le 4. \end{aligned}$$

Variable 1 earns more per unit of the shared limit. Set it to 4, then use the remaining unit for variable 2.

$$x = (4, 1), \qquad z = 3(4) + 2(1) = 14.$$

The upper bound has decreased from 20 to 14. We must still test the omitted second row.

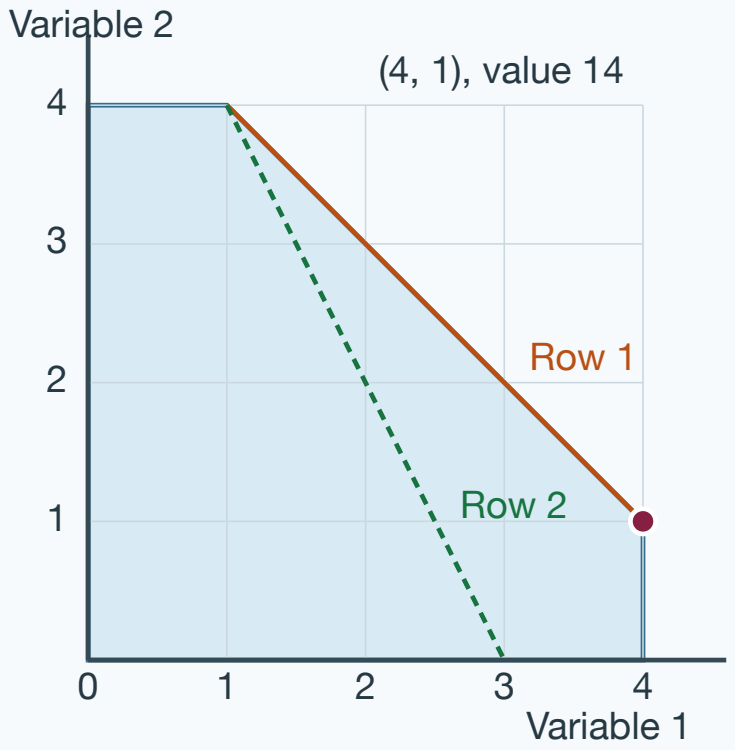
The smaller region still contains an infeasible optimum

Current solution: $x = (4, 1)$, value 14.

> Check the remaining row:
> $2x_1 + x_2 = 2(4) + 1 = 9 > 6.$

Add $2x_1 + x_2 \leq 6$. It removes the current point while preserving every feasible point of the full LP.

The orange line is enforced; the dashed green line is not yet enforced.



Horizontal: x_1; vertical: x_2. Orange: $x_1 + x_2 = 5$. Green: $2x_1 + x_2 = 6$.

After adding row 2, solve the complete LP

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5, \quad 2x_1 + x_2 \le 6, \\ & 0 \le x_1 \le 4, \quad 0 \le x_2 \le 4. \end{aligned}$$

The two added boundaries meet where $x_1 + x_2 = 5$ and $2x_1 + x_2 = 6$. Subtracting gives $x_1 = 1$, then $x_2 = 4$.

$$x = (1, 4), \qquad z = 3(1) + 2(4) = 11.$$

All original constraints are satisfied. The next page verifies optimality without relying only on the plot.

The current upper bound is now attained by a feasible point

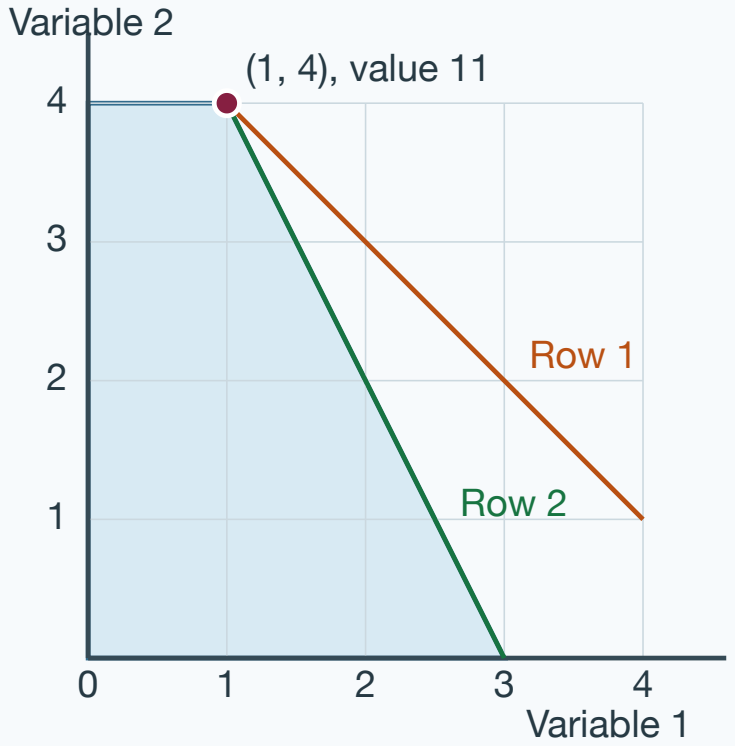
$$(x_1 + x_2) + (2x_1 + x_2) \leq 5 + 6.$$

$$3x_1 + 2x_2 \leq 11.$$

The point $(1, 4)$ is feasible and has value 11. It attains this upper bound.

The complete LP optimum is 11.

A stopping certificate checks feasibility for the full model, not just the constraints already drawn.

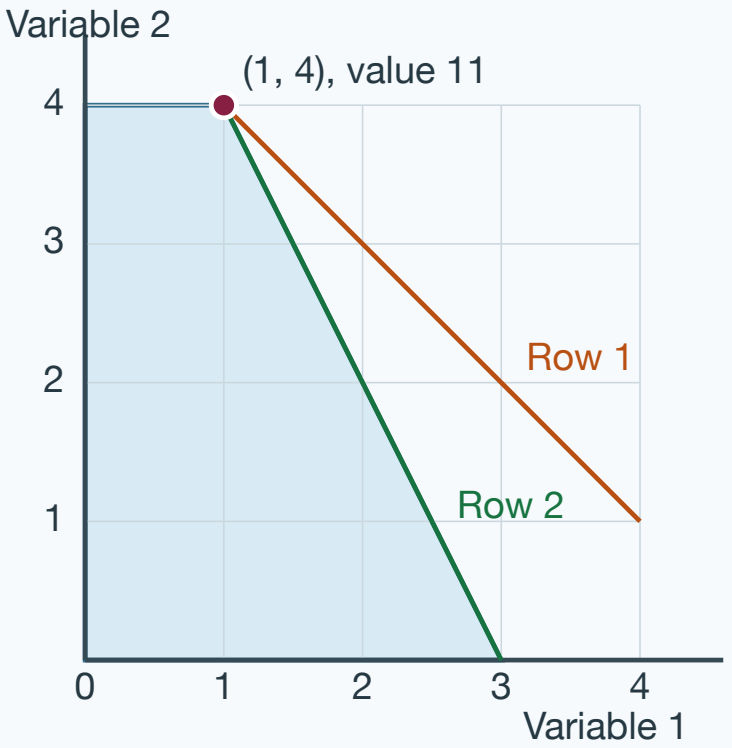


Horizontal: x_1; vertical: x_2. Orange: $x_1 + x_2 = 5$. Green: $2x_1 + x_2 = 6$.

Compare the three restricted problems

Rows added	Best point	Upper bound
None	(4,4)	20
Row 1	(4,1)	14
Rows 1 and 2	(1,4)	11

The upper bounds decrease. The final point satisfies every original row and attains the final bound.



Horizontal: x_1; vertical: x_2. Orange: $x_1 + x_2 = 5$. Green: $2x_1 + x_2 = 6$.

Separation: find a violated original constraint

Let x^* be an **optimal solution of the current restricted LP**. Check an omitted constraint from the original LP:

$$a_i^\top x \le b_i \qquad \text{is violated when} \qquad a_i^\top x^* > b_i.$$

Searching for such a constraint is called **separation**.

A violated constraint exists: add it and solve the restricted LP again.

No omitted constraint is violated: x^* is feasible—and optimal—for the original LP.

An unbounded restricted LP needs separate treatment: this test assumes an optimal solution exists.

With finitely many original constraints, adding a new one each time permits only finitely many additions.

Check the direction of the bound

We maximize the **same objective** in the original LP (500 constraints) and a restricted LP (20 retained constraints). Let x^* be an optimal solution of the restricted LP, with objective value 80.

1. Can the original LP have a feasible solution of value 85?
2. Suppose x^* satisfies all 500 constraints. What is the original LP's optimum?
3. Suppose x^* violates an omitted constraint. Is 80 still an upper bound for the original LP?

Explain using this fact: every point satisfying all 500 original constraints also satisfies the 20 retained constraints. The converse need not hold.

Pause for 4 minutes, then check the next page.

Fewer constraints give a larger feasible set

The objective is unchanged. The LP with 20 retained constraints has optimal solution x^* and value 80.

1. No. Every point satisfying all 500 constraints is also allowed by the 20 retained constraints, where the maximum is 80. Its value cannot be 85.

2. The original optimum is 80. If x^* satisfies all 500 constraints, it is feasible for the original LP and attains the upper bound 80.

3. Yes. Every original feasible point still has value at most 80. The violation makes x^* infeasible for the original LP; it does not invalidate that upper bound.

Columns and constraints: two different stopping checks

	Column generation here	Constraint generation here
Smaller problem	Min LP with fewer variables	Max LP with fewer constraints
Restricted solution	Feasible for the full primal	May violate omitted rows
What to check	All omitted dual inequalities	All omitted primal inequalities
What to add	A negative-reduced-cost column	A violated original row
At termination	Full dual certificate	Full primal feasibility

The bound direction follows from the objective sense and whether we restricted variables or removed constraints; do not memorize it without those details.

How does this relate to cutting-plane methods?

In **constraint generation**, we delay original constraints, then add violated ones. Every full-feasible solution satisfies each added row.

More generally, a **cutting-plane method** adds a valid inequality that removes the current point while preserving the feasible solutions of interest.

Our geometric example uses only original LP constraints. No integer-programming cuts are needed.

The important computational question is how to find a useful violated inequality efficiently.

What to take away

Column generation: solve a restricted LP, read its dual solution, price missing columns, add a column, and repeat until every reduced cost has the required sign.

Constraint generation: solve a relaxation, separate omitted rows, add a violated row, and repeat until the solution is feasible for the full LP.

Both methods need a complete stopping test. Searching a convenient subset is not enough unless it covers every possible violation.

Read BT §§6.1–6.3. Appendix: proofs and optional decomposition connections.

Appendix: proofs and optional connections

The first pages justify the stopping tests and complete the arithmetic for the cutting-stock example.

- Column-generation optimality proof
- Why the pattern list covers every case
- Verify both reoptimizations
- Constraint-generation optimality proof
- Optional: Dantzig–Wolfe and Lagrangian connections

Why the column-generation stopping test proves optimality

Consider $\min\{\sum_j c_j x_j : \sum_j a_j x_j \geq b,\ x \geq 0\}$. Let x^J, p^J be an optimal primal–dual pair for a feasible, bounded restricted master.

Extend x^J by zeros for omitted variables. It remains full-primal feasible. If pricing finds $a_j^\top p^J \leq c_j$ for every column, then p^J is full-dual feasible.

$$\underbrace{b^\top p^J}_{\text{full lower bound}} \quad \leq \quad z^* \quad \leq \quad \underbrace{\sum_{j \in J} c_j x_j^J}_{\text{full feasible value}} \quad .$$

Restricted strong duality makes the two ends equal. Therefore both full solutions are optimal.

If the starting restricted master is infeasible, construct feasible starting columns or use a Phase I master; do not read dual prices from a nonexistent optimum.

Enumerate the maximal cutting patterns

A pattern can use at most one width-6 piece, since $2(6) > 10$.

Case	Patterns that cannot accept another piece
One width-6 piece	(1,0,1) or (0,1,1)
No width-6 piece; two width-4 pieces	(0,2,0)
No width-6 piece; one width-4 piece	(2,1,0)
No width-6 or width-4 pieces	(3,0,0)

> Every smaller feasible pattern is componentwise no larger than one of these five. With $p \geq 0$, its value cannot exceed that larger pattern's value.

The initial pattern (0,0,1) is feasible but not maximal. Keeping it as an initial column is harmless; pricing also checks the patterns that fill its remaining space.

Verify the first reoptimization with a matching bound

After adding pattern 4, use $x = (4/3, 1/2, 0, 2)$ and $p = (1/3, 1/2, 1/2)$.

$$3p_1 = 1, \quad 2p_2 = 1, \quad p_3 = \frac{1}{2} \le 1, \quad p_2 + p_3 = 1.$$

$$4p_1 + 3p_2 + 2p_3 = \frac{4}{3} + \frac{3}{2} + 1 = \frac{23}{6} = \sum_{j=1}^{4} x_j.$$

Primal feasibility was checked in the main sequence; these inequalities verify restricted dual feasibility. Equal objectives prove restricted optimality.

It is not yet a full certificate: pattern 5 has $2p_1 + p_2 = 7/6 > 1$.

Verify the second reoptimization and the whole-roll distinction

After adding pattern 5, $x = (2/3, 0, 0, 2, 1)$ produces exactly $(4, 3, 2)$, with objective $11/3$.

$$p = \left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right), \quad (p^\top a^{(1)}, \dots, p^\top a^{(6)}) = \left(1, \frac{2}{3}, \frac{2}{3}, 1, 1, 1\right).$$

> This proves the continuous LP optimum is $11/3$. It does not permit us to physically cut a fraction of a roll.

For comparison, one copy of pattern 1, two of pattern 4, and one of pattern 5 use 4 whole rolls and produce $(5, 3, 2)$. Since any whole-roll plan needs at least $\lceil 11/3 \rceil = 4$ rolls, this plan is also best among whole-roll plans in this small example.

Why constraint generation proves optimality

Let P be the full feasible set and P_R the set defined by the retained rows. Then $P \subseteq P_R$. Assume $\hat{x}$ maximizes $c^\top x$ over P_R.

$$c^\top x \le c^\top \hat{x} \qquad \text{for every } x \in P.$$

If separation certifies $\hat{x} \in P$, then this upper bound is attained within P. Hence $\hat{x}$ is a full optimum.

Adding any violated original row preserves P and removes the current infeasible point. With finitely many rows, never discard added rows and never add a duplicate; only finitely many such additions are possible.

For a minimization problem with omitted constraints, the restricted optimum is instead a lower bound.

Optional: replace points by combinations of columns

Suppose a bounded polyhedron Q has vertices $v^1, \dots, v^K$. Every $x \in Q$ can be written as

$$x = \sum_{k=1}^{K} \lambda_k v^k, \qquad \sum_{k=1}^{K} \lambda_k = 1, \qquad \lambda_k \geq 0.$$

> Example: $Q = \{x \geq 0 : x_1 + x_2 \leq 2\}$ has vertices $(0,0), (2,0), (0,2)$. The point $(0,1)$ is $\frac{1}{2}(0,0) + \frac{1}{2}(0,2)$.

Dantzig–Wolfe decomposition uses such combinations to keep the constraints defining Q together and put the remaining constraints in a master LP. Those remaining constraints are called linking constraints.

Optional: the master LP and its pricing problem

Start with $\min\{c^\top x : Dx \geq d,\ x \in Q\}$, where Q is a nonempty bounded polyhedron. Substitution gives

$$\min_{\lambda \geq 0} \sum_k (c^\top v^k)\lambda_k \quad \text{s.t.} \quad \sum_k (Dv^k)\lambda_k \geq d, \quad \sum_k \lambda_k = 1.$$

Use $p \geq 0$ for the linking rows and unrestricted α for the sum-to-one row. Column k has reduced cost

$$c^\top v^k - p^\top D v^k - \alpha.$$

> Find the smallest reduced cost by solving $\min_{x \in Q} (c - D^\top p)^\top x - \alpha$. A linear objective on bounded Q attains its minimum at a vertex.

Optional: the same subproblem gives a Lagrangian bound

For $\min\{c^\top x : Dx \geq d,\ x \in Q\}$, choose $p \geq 0$ and define

$$L(x, p) = c^\top x + p^\top (d - Dx).$$

For every original feasible x, $d - Dx \leq 0$, so $L(x, p) \leq c^\top x$. Consequently,

$$p^\top d + \min_{x \in Q} (c - D^\top p)^\top x \ \leq\ \text{original minimum}.$$

This lower-bound calculation uses the same optimization over Q as column pricing. It explains the connection between continuous-LP decomposition and relaxing linking constraints.