Applications of Duality: Revenue Management and Robust Optimization

How many tickets should an airline sell for each journey? How much should a factory produce when resource use is uncertain?

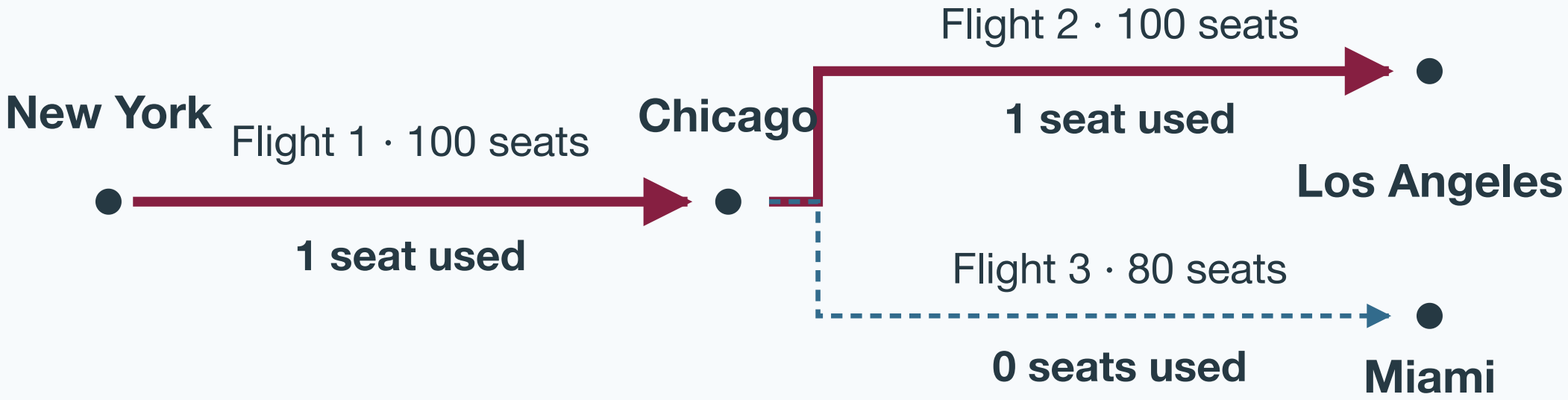
> **Two applications:** maximize airline ticket revenue; choose production quantities that satisfy the constraints for every data case we allow.

We will write each model, verify its numbers, and interpret its dual variables.

- Revenue management
- Robust optimization
- Uncertain capacities
- Proofs and further applications

One passenger may need seats on two flights

An airline has three direct flights with limited seats. Each direct flight is a **flight leg**; a passenger's complete journey is an **itinerary**.



Shown: one New York → Chicago → Los Angeles passenger uses one seat on each of flights 1 and 2.

> A connecting passenger uses one seat on **each** flight in the journey. Those seats could instead carry two direct-flight passengers.

The highest ticket fare need not give the most total revenue

Let x_j be the number of tickets sold for journey j. **Fare** means the ticket price per passenger for the complete journey.

j	Complete journey	Flights	Fare	Demand limit
1	New York → Chicago	1	$200	50
2	New York → Chicago → Los Angeles	1, 2	$400	100
3	New York → Chicago → Miami	1, 3	$350	70
4	Chicago → Los Angeles	2	$250	60
5	Chicago → Miami	3	$180	90

Demand limit: the maximum tickets we assume can be sold at that price.

> **One seat on each of flights 1 and 2:** one connecting ticket earns **$400**; two direct tickets earn **$200 + $250 = $450**, if both direct passengers want tickets.

Direct-ticket sales are demand limited. Choose the quantities of all five journeys together.

Choose ticket quantities subject to seat and demand limits

x_1, x_2, x_3: New York to Chicago, Los Angeles, Miami; the last two connect in Chicago. x_4, x_5: Chicago to Los Angeles, Miami.

$$
\begin{aligned}
\max \quad & 200x_1 + 400x_2 + 350x_3 + 250x_4 + 180x_5 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 \le 100 && \text{New York} \to \text{Chicago}, \\
& x_2 + x_4 \le 100 && \text{Chicago} \to \text{Los Angeles}, \\
& x_3 + x_5 \le 80 && \text{Chicago} \to \text{Miami}, \\
& 0 \le x \le (50, 100, 70, 60, 90).
\end{aligned}
$$

Each x_j is the number of tickets sold. Its upper bound is the demand limit for that journey. This LP treats these quantities as continuous.

> The objective is total ticket revenue in dollars. Each flight constraint counts the passengers occupying its seats.

A feasible allocation earns $57,100

Itinerary j	1	2	3	4	5
Accepted quantity	50	40	10	60	70
Demand limit	50	100	70	60	90

$$\begin{aligned}\text{Leg 1: } & 50 + 40 + 10 = 100,\\ \text{Leg 2: } & 40 + 60 = 100, \quad \text{Leg 3: } 10 + 70 = 80,\\ \text{Revenue: } & 200(50) + 400(40) + 350(10) + 250(60) + 180(70) = 57{,}100.\end{aligned}$$

This proves feasibility and computes the revenue. Optimality requires an upper bound of the same value.

The flight LP and its dual

x_j: tickets sold for journey j. Dual variables: p_i for flight i's seat limit; u_j for journey j's demand limit.

Primal · maximize ticket revenue

$$\begin{aligned}\max \quad & 200x_1 + 400x_2 + 350x_3 \\ & +250x_4 + 180x_5 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \le 100 \quad (1), \\ & x_2 + x_4 \le 100 \quad (2), \\ & x_3 + x_5 \le 80 \quad (3), \\ & 0 \le x \le (50, 100, 70, 60, 90).\end{aligned}$$

(1) New York → Chicago
(2) Chicago → Los Angeles
(3) Chicago → Miami

Dual · minimize the revenue bound

$$\begin{aligned}\min \quad & 100p_1 + 100p_2 + 80p_3 \\ & +50u_1 + 100u_2 + 70u_3 \\ & +60u_4 + 90u_5 \\ \text{s.t.} \quad & p_1 + u_1 \ge 200, \\ & p_1 + p_2 + u_2 \ge 400, \\ & p_1 + p_3 + u_3 \ge 350, \\ & p_2 + u_4 \ge 250, \\ & p_3 + u_5 \ge 180, \\ & p_1, p_2, p_3, u_1, \dots, u_5 \ge 0.\end{aligned}$$

For x_2, flights 1 and 2 and its demand limit give $p_1 + p_2 + u_2 \ge 400$.

Use complementary slackness to find a dual solution

Dual variables: p_i for flight i's seat limit; u_j for journey j's demand limit.

Positive primal slack		Required: zero dual variable
New York → Chicago → Los Angeles	$x_2 = 40 < 100$	$u_2 = 0$
New York → Chicago → Miami	$x_3 = 10 < 70$	$u_3 = 0$
Chicago → Miami	$x_5 = 70 < 90$	$u_5 = 0$
Positive primal variable		**Required: active dual constraint**
New York → Chicago	$x_1 = 50 > 0$	$p_1 + u_1 = 200$
New York → Chicago → Los Angeles	$x_2 = 40 > 0$	$p_1 + p_2 + u_2 = 400$
New York → Chicago → Miami	$x_3 = 10 > 0$	$p_1 + p_3 + u_3 = 350$
Chicago → Los Angeles	$x_4 = 60 > 0$	$p_2 + u_4 = 250$
Chicago → Miami	$x_5 = 70 > 0$	$p_3 + u_5 = 180$

$$p_3 = 180, \quad p_1 = 350 - 180 = 170, \quad p_2 = 400 - 170 = 230,$$
$$u_1 = 200 - 170 = 30, \qquad u_4 = 250 - 230 = 20.$$

Next: check dual feasibility and the matching objective value.

Check feasibility and the matching upper bound

Itinerary	Dual left side	Fare
1	170 + 30 = 200	200
2	170 + 230 = 400	400
3	170 + 180 = 350	350
4	230 + 20 = 250	250
5	180	180

$$100(170) + 100(230) + 80(180) + 50(30) + 60(20) = 57{,}100.$$

All dual variables are nonnegative, all five constraints hold, and the dual value equals the feasible revenue. Both solutions are optimal.

How much revenue must we give up to reserve one seat?

Before selling tickets, **reserve one New York → Chicago seat and leave it empty**. Only 99 of its 100 seats remain for the original journeys. Other flight capacities stay at 100 to Los Angeles and 80 to Miami.

Original optimal revenue: $57,100. Compare these ticket quantities; all fares and demand limits stay fixed:

Journey · fare	Original	Seat reserved	Demand limit
New York → Chicago · $200	50	50	50
New York → Chicago → Los Angeles · $400	40	40	100
New York → Chicago → Miami · $350	10	9	70
Chicago → Los Angeles · $250	60	60	60
Chicago → Miami · $180	70	71	90

One fewer connecting ticket frees two seats. The extra Chicago → Miami ticket fills the second seat again. Only the New York → Chicago seat stays empty.

> **Sell one fewer direct ticket:** lose $200. **Use the change above:** lose $350 – $180 = **$170**, while keeping all 50 direct customers paying $200.

Why is $170 the smallest possible revenue loss?

Dual solution: $p = (170, 230, 180)$ (flight capacities); $u = (30, 0, 0, 20, 0)$ (demand limits).

Fares and routes are unchanged, so **the dual constraints still hold**. Only New York → Chicago capacity changes: **100 → 99**. Weak duality bounds the original ticket revenue:

$$\begin{aligned} \text{Original bound} &= 100(170) + 100(230) + 80(180) \\ &\quad +50(30) + 60(20) = 57{,}100, \\ \text{One seat reserved} &= 99(170) + 100(230) + 80(180) \\ &\quad +50(30) + 60(20) = 56{,}930. \end{aligned}$$

Evidence	What it proves
One fewer $350 New York → Miami ticket; one more $180 Chicago → Miami ticket	We can earn $57,100 – $350 + $180 = $56,930.
Dual upper bound	No feasible ticket quantities can earn more than $56,930.

Smallest revenue loss: $57{,}100 - 56{,}930 = 170 = p_1$.

Now a new customer offers $180 for the reserved seat

New offer: one additional customer pays **$180 for New York → Chicago**. All 50 original customers paying $200 remain included; the original fares and demand limits stay fixed.

Case	Total ticket revenue
Keep the original optimal quantities; decline the new offer	$57,100
Reserve one New York → Chicago seat; do not sell it yet	$56,930
Use the reserved seat for the new $180 customer	$56,930 + $180 = $57,110

The new customer fills the empty New York → Chicago seat. The other flights remain unchanged: 100 passengers to Los Angeles and 80 to Miami.

$$\text{New payment} - \text{revenue given up} = 180 - 170 = 10.$$

Accept: the $180 payment exceeds the smallest revenue loss of $170. We gain **$10**.

Optional follow-up: a new passenger uses two flights

New problem: what if production takes longer?

We now consider a factory production problem. The factory has **8 machine-hours**. Estimated times are 2 hours per batch of product 1 and 1 hour per batch of product 2.

Suppose we choose **3 batches of product 1** and **2 batches of product 2**.

Product 1: hours/batch	Product 2: hours/batch	Total machine-hours	Within 8 hours?
2 (estimated)	1 (estimated)	$2(3) + 1(2) = 8$	Yes
3 (slower)	1	$3(3) + 1(2) = 11$	No

Decision: choose the production quantities before processing times are known. Which quantities satisfy the capacity constraint for **every processing-time combination included in our model**?

Define the production quantities and uncertain times

x_1, x_2: batches of products 1 and 2. Profit is \$3 and \$2 per batch, respectively; at most 5 batches can be produced in total.

z_1, z_2: additional processing hours per batch. **Assume** $z_1, z_2 \geq 0$ and $z_1 + z_2 \leq 1$. Each allowed pair is a **scenario**; all these pairs form the **uncertainty set**.

$$\begin{aligned} \max_{x_1, x_2} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & (2 + z_1)x_1 + (1 + z_2)x_2 \leq 8 \\ & \qquad \text{for every allowed } (z_1, z_2), \\ & x_1 + x_2 \leq 5, \qquad x_1, x_2 \geq 0. \end{aligned}$$

Robust feasibility: the *same* quantities x_1, x_2 satisfy the constraints for every allowed pair. We choose x; we do not choose the actual processing-time increases z.

Check two choices of production quantities

Factory: 8 machine-hours, at most 5 batches, profit $3x_1 + 2x_2$. Processing times are $2 + z_1$ and $1 + z_2$ hours per batch, with $z_1, z_2 \geq 0$, $z_1 + z_2 \leq 1$.

Increases (z_1, z_2)	Hours for $x_1 = 3, x_2 = 2$	Hours for $x_1 = 2, x_2 = 2$
$(0, 0)$	$2(3) + 1(2) = 8$ ✓	$2(2) + 1(2) = 6$ ✓
$(1, 0)$	$3(3) + 1(2) = 11$ ✗	$3(2) + 1(2) = 8$ ✓
$(0, 1)$	$2(3) + 2(2) = 10$ ✗	$2(2) + 2(2) = 8$ ✓

✓ satisfies the 8-hour constraint; ✗ violates it. Both choices satisfy the 5-batch limit.

Profit falls from $3(3) + 2(2) = 13$ to $3(2) + 2(2) = 10$, but $(2, 2)$ passes all three checks. Next: why do these checks cover *every* allowed pair?

Why these three cases cover every allowed pair

For the factory, processing-time increases satisfy $z_1, z_2 \geq 0$, $z_1 + z_2 \leq 1$. Every such pair is a convex combination of $(0,0), (1,0), (0,1)$.

Machine-hour use is linear in these increases. If it is at most 8 at all three vertices, it is at most 8 for every allowed pair.

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 8 \quad \text{case } (1,0), \\ & 2x_1 + 2x_2 \leq 8 \quad \text{case } (0,1), \\ & x_1 + x_2 \leq 5, \quad x_1, x_2 \geq 0. \end{aligned}$$

The estimated-time row $2x_1 + x_2 \leq 8$ is implied by these rows and nonnegativity.

> Solving this LP gives $x_1 = x_2 = 2$ batches and profit \$10.

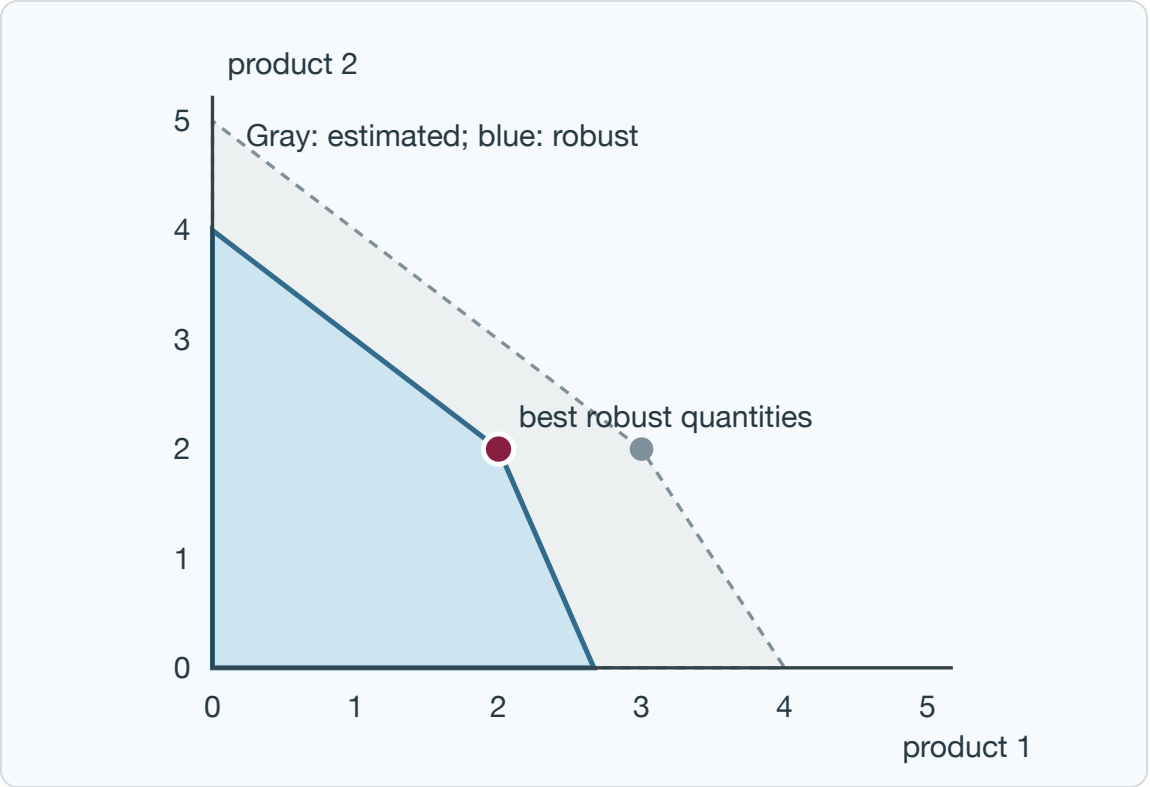
How the allowed time increases affect production

Factory: $\max 3x_1 + 2x_2$, at most 5 batches and 8 machine-hours. x_i is batches of product i.

$$(2 + z_1)x_1 + (1 + z_2)x_2 \leq 8$$

for every $0 \leq z_1, z_2 \leq 1$ with $z_1 + z_2 \leq \Gamma$.

The **uncertainty budget** Γ is the allowed total increase. At $\Gamma = 0$, use estimated times. At $\Gamma = 1$, recover our example. At $\Gamma = 2$, both times may increase by 1.



Budget 1: product 1 = 2 batches; product 2 = 2 batches; profit $10.

For chosen quantities, find the largest additional hours

Return to $z_1, z_2 \geq 0,\ \ z_1 + z_2 \leq 1$. For chosen batches x_1, x_2, total machine-hour use is

$$(2 + z_1)x_1 + (1 + z_2)x_2 = \underbrace{2x_1 + x_2}_{\text{estimated hours}} + \underbrace{x_1 z_1 + x_2 z_2}_{\text{additional hours}}.$$

Hold x_1, x_2 fixed. To check their feasibility, find the largest additional hours by solving this LP **over** z:

$$\begin{aligned} \max_{z_1, z_2} \quad & x_1 z_1 + x_2 z_2 \\ \text{s.t.} \quad & z_1 + z_2 \leq 1, \\ & z_1, z_2 \geq 0. \end{aligned}$$

The maximum is $\max\{x_1, x_2\}$: give the full allowed increase to the product with more batches. For $x_1 = 3, x_2 = 2$, it is 3 hours; total use can reach $8 + 3 = 11$.

The dual bounds additional machine-hours

Hold batches x_1, x_2 fixed. The dual variable $q \geq 0$ corresponds to $z_1 + z_2 \leq 1$.

Primal · largest additional hours

$$\begin{aligned} \max_{z_1, z_2} \quad & x_1 z_1 + x_2 z_2 \\ \text{s.t.} \quad & z_1 + z_2 \leq 1, \\ & z_1, z_2 \geq 0. \end{aligned}$$

Dual · smallest upper bound

$$\begin{aligned} \min_{q} \quad & q \\ \text{s.t.} \quad & q \geq x_1, \\ & q \geq x_2, \qquad q \geq 0. \end{aligned}$$

Strong duality: the minimum on the right equals the largest additional hours on the left.

Choose q satisfying the dual constraints. Estimated hours plus this bound must fit within 8 hours:

$$2x_1 + x_2 + q \leq 8.$$

Check: for $x_1 = 3, x_2 = 2$, the smallest q is 3, and $8 + 3 = 11 > 8$.

The general uncertain-consumption LP and its dual

For chosen quantities x, require consumption $a^\top x \le \beta$ for every coefficient vector a in the nonempty set $U = \{a : Da \le d\}$. Here β is the resource capacity.

Primal · largest consumption

$$\begin{aligned} \max_{a} \quad & a^\top x \\ \text{s.t.} \quad & Da \le d, \\ & a \text{ unrestricted in sign.} \end{aligned}$$

Dual · smallest upper bound

$$\begin{aligned} \min_{\lambda} \quad & d^\top \lambda \\ \text{s.t.} \quad & D^\top \lambda = x, \\ & \lambda \ge 0. \end{aligned}$$

There is one dual variable λ_i per row of $Da \le d$. Any bounds on a are included in those rows; without separate sign restrictions on a, the dual constraints are equalities.

By strong duality, the capacity limit holds for every $a \in U$ exactly when some λ satisfies

$$D^\top \lambda = x, \qquad d^\top \lambda \le \beta, \qquad \lambda \ge 0.$$

Why the new dual variables certify every scenario

Suppose $D^\top \lambda = x$, $d^\top \lambda \le \beta$, and $\lambda \ge 0$. For any a with $Da \le d$,

$$a^\top x = a^\top D^\top \lambda = (Da)^\top \lambda \le d^\top \lambda \le \beta.$$

One feasible dual certificate bounds every allowed scenario at once. Strong duality supplies the reverse implication.

Full equivalence proof in the appendix.

Uncertain capacities require a different worst-case direction

$$\max\ 40x_1 + 30x_2 \quad \text{s.t. } 2x_1 + x_2 \le b_1, \quad x_1 + 2x_2 \le b_2, \quad x \ge 0.$$

The capacities satisfy $80 \le b_1 \le 100$, $60 \le b_2 \le 80$, and $b_1 + b_2 \ge 150$.

> A fixed production plan must work for every allowed capacity pair. Smaller capacities make these $\le$ constraints harder to satisfy.

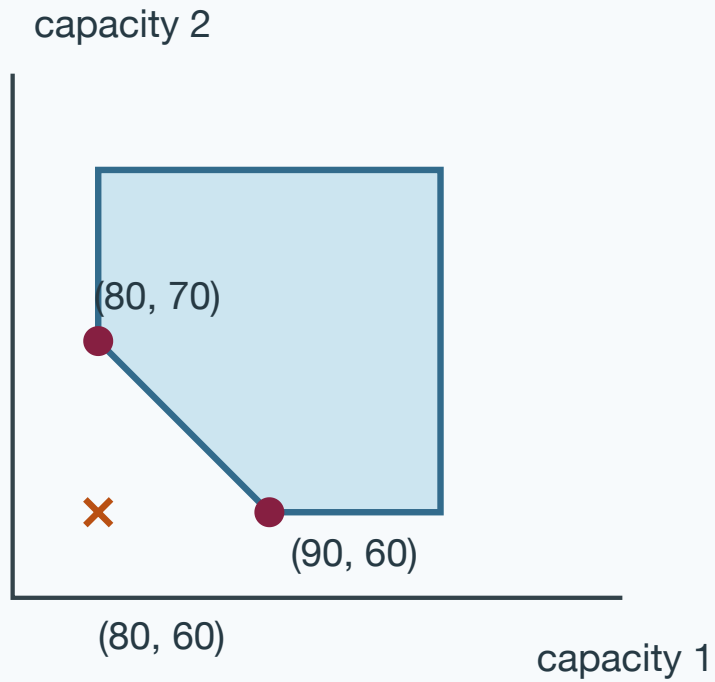
Therefore row i requires $a_i^\top x \le \min_{b \in U} b_i$.

Minimize each right-hand side separately

For row 1, the minimum is $b_1 = 80$, attained at $(80, 70)$. For row 2, the minimum is $b_2 = 60$, attained at $(90, 60)$.

$$2x_1 + x_2 \le 80, \qquad x_1 + 2x_2 \le 60.$$

A plan must satisfy row 1 in its worst scenario and row 2 in its worst scenario. Those scenarios need not be identical.



The two row minima need not occur in the same scenario.

Solve the nominal and robust production problems

Model	Capacities	Optimal quantities	Profit
Nominal	$(100, 80)$	$(40, 20)$	2200
Robust	$(80, 60)$	$(100/3, 40/3)$	$5200/3 \approx 1733.33$

At the robust optimum, solve $2x_1 + x_2 = 80$ and $x_1 + 2x_2 = 60$. The nonnegative dual variables are $(50/3, 20/3)$.

$$80(50/3) + 60(20/3) = 5200/3.$$

The feasible production plan attains its dual upper bound. Protection reduces the best profit by about 466.67.

Correlation does not help this fixed-plan RHS model

The pair $(80, 60)$ is excluded by $b_1 + b_2 \geq 150$. Nevertheless, the separate minima remain 80 and 60.

> Removing that pair does not change the robust feasible set: each row must still withstand its own minimum.

Correlation matters when it changes a row's worst value, or when uncertain coefficients jointly enter the same constraint. It does not automatically improve every robust model.

Allowing decisions to change after observing the scenario would be a different model.

A shared RHS budget saturates at one full deviation

Let $b_i = \bar{b}_i - \hat{b}_i z_i$, where $0 \le z_i \le 1$, $\sum_i z_i \le \Gamma$, and $\Gamma \ge 0$.

For one row i, the adversary can concentrate its budget there: $\max z_i = \min\{1, \Gamma\}$.

$$a_i^\top x \le \bar{b}_i - \hat{b}_i \min\{1, \Gamma\}.$$

For $\Gamma \ge 1$, every row already has to survive its full individual deviation. Increasing the budget further does not change this fixed-plan RHS model.

Calculate the budget effect rather than assuming it

Use $\bar{b} = (100, 80)$, $\hat{b} = (20, 20)$ in the production LP $\max 40x_1 + 30x_2$.

Γ	Robust capacities	Optimal quantities	Profit
0	(100, 80)	(40, 20)	2200
0.5	(90, 70)	$(110/3, 50/3)$	$5900/3$
1	(80, 60)	$(100/3, 40/3)$	$5200/3$
1.5 or 2	(80, 60)	$(100/3, 40/3)$	$5200/3$

Budgeted coefficient uncertainty can continue changing after Γ=1. Pure RHS uncertainty in this fixed-plan form cannot.

The uncertain quantity determines the inner LP

Uncertainty	Worst-case test	Duality gives
Coefficients in a ≤ row	$\max_{a \in U} a^\top x \leq \beta$	An upper bound on consumption.
RHS in a ≤ row	$a^\top x \leq \min_{b \in U} b_i$	A lower bound on available capacity.
Returns in a maximized objective	$\max_x \min_{r \in U} r^\top x$	A guaranteed return for the chosen portfolio.

Full RHS dual construction, including its signs.

Portfolio and routing examples.

Interpret a dual solution; then use duality as a modeling tool

Revenue management: use capacity dual variables to evaluate an additional ticket, and verify the changes in ticket quantities and total revenue.

Robust optimization: fix a decision, optimize over the uncertainty set, write the inner dual, and substitute its finite constraints.

Keep units, sign conventions, decision timing, and the uncertainty set explicit.

Next, large-scale LP uses dual variables to find improving columns and violated constraints. Cutting-stock column generation belongs to that next resource.

Appendix: complete derivations and additional applications

These pages complete the derivations and retain further continuous-LP applications.

- Revenue dual and marginal rules
- Robust equivalence proofs
- RHS uncertainty signs
- Portfolio and corridor models
- Dispatch and capacity value

Derive the general booking dual

Let $A_{ij} = 1$ when itinerary j uses leg i, and 0 otherwise. Let fares be f, capacities C, and demand limits d.

$$\begin{aligned} \max \quad & f^\top x \\ \text{s.t.} \quad & Ax \le C, \\ & 0 \le x \le d. \end{aligned} \qquad \begin{aligned} \min \quad & C^\top p + d^\top u \\ \text{s.t.} \quad & A^\top p + u \ge f, \\ & p, u \ge 0. \end{aligned}$$

For any feasible pair, $f^\top x \le p^\top A x + u^\top x \le p^\top C + u^\top d$.

> The max primal has ≤ rows, so the min dual has nonnegative variables. Nonnegative primal quantities give ≥ dual constraints.

Every complementary pair in the booking example

Primal $x = (50, 40, 10, 60, 70)$; dual $p = (170, 230, 180)$, $u = (30, 0, 0, 20, 0)$.

$$C - Ax = (0, 0, 0), \qquad d - x = (0, 60, 60, 0, 20).$$

$$A^\top p + u - f = (0, 0, 0, 0, 0).$$

$$p_i(C_i - A_i x) = 0, \qquad u_j(d_j - x_j) = 0, \qquad x_j((A^\top p)_j + u_j - f_j) = 0.$$

Positive primal slack forces the paired dual variable to zero. Zero primal slack merely permits a nonzero dual variable.

Recover the allocation directly from the certificates

The positive dual variables p_1, p_2, p_3, u_1, u_4 require the corresponding primal rows to be active:

$$x_1 = 50, \quad x_4 = 60, \quad x_2 + x_4 = 100, \quad x_1 + x_2 + x_3 = 100, \quad x_3 + x_5 = 80.$$

$$x_2 = 40, \qquad x_3 = 10, \qquad x_5 = 70.$$

> Check every demand bound and nonnegativity afterwards. Complementary equations alone do not replace feasibility checks.

The same booking basis gives a precise validity region

Keep the same demand limits and the active bounds $x_1 = 50, x_4 = 60$. For capacities $C = (C_1, C_2, C_3)$, solve the three active leg equations:

$$x_2 = C_2 - 60, \qquad x_3 = C_1 - C_2 + 10, \qquad x_5 = C_3 - C_1 + C_2 - 10.$$

$$\begin{aligned} 0 \le C_2 - 60 \le 100, \\ 0 \le C_1 - C_2 + 10 \le 70, \\ 0 \le C_3 - C_1 + C_2 - 10 \le 90. \end{aligned}$$

Within this region, revenue equals $170C_1 + 230C_2 + 180C_3 + 2700$. Outside it, reoptimize.

A bid price is the reduced-cost test for a new request

A flight's **bid price** means its capacity dual variable. For a proposed new ticket, add these values over the flights it uses.

Add a new nonnegative variable t, fare r, and resource-use column a. The old solution with $t = 0$ remains feasible.

$$\bar{r} = r - p^{\top} a.$$

> For this maximization problem, $\bar{r} \leq 0$ preserves the old dual bound. If $\bar{r} > 0$, the old certificate fails for the new column and primal simplex can test an improving move.

A positive reduced profit is a direction test; degeneracy or capacity restrictions can limit its feasible step.

A connecting passenger uses two flights

Separate case: start again from the original **$57,100** optimum. A new passenger offers **$390 for New York → Chicago → Los Angeles**.

Flight	Capacity dual variable	Seats left for original journeys
New York → Chicago	$p_1 = 170$	100 → 99
Chicago → Los Angeles	$p_2 = 230$	100 → 99

The same dual variables remain feasible. With all other limits unchanged, reducing these two seat limits by 1 decreases the dual upper bound by $170 + $230:

$$\begin{aligned}\text{Revenue from original tickets} &\le 57{,}100 - 170 - 230 = 56{,}700,\\ \text{Total revenue with new ticket} &\le 56{,}700 + 390 = 57{,}090.\end{aligned}$$

Decline: total revenue would be at most $57,090, below the original $57,100.

Return to the one-seat example

Robust coefficient rows: prove both directions

Let $U = \{a : Da \leq d\} \neq \varnothing$. Fix x and a finite bound β.

If there is $\lambda \geq 0$ with $D^\top \lambda = x$ and $d^\top \lambda \leq \beta$, then for every $a \in U$, $a^\top x = (Da)^\top \lambda \leq d^\top \lambda \leq \beta$.

Conversely, if $a^\top x \leq \beta$ for every $a \in U$, the feasible inner LP has a finite optimum. LP strong duality gives an optimal $\lambda \geq 0$ with $D^\top \lambda = x$ and $d^\top \lambda \leq \beta$.

The nonempty uncertainty-set assumption avoids treating an impossible scenario model as a useful guarantee.

Budgeted coefficient uncertainty with individual caps

Let $a = \bar{a} + \hat{a} \odot z$, with $\hat{a}_j \geq 0$, $0 \leq z_j \leq 1$, $\sum_j z_j \leq \Gamma$, and $x \geq 0$. Here $\odot$ means componentwise multiplication.

$$\max_{z} \sum_{j} \hat{a}_j x_j z_j \quad \text{s.t.} \ \sum_{j} z_j \leq \Gamma, \quad 0 \leq z_j \leq 1.$$

Use $\lambda \geq 0$ for the budget row and $\mu_j \geq 0$ for each upper bound.

$$\min \ \Gamma\lambda + \sum_{j} \mu_j \quad \text{s.t.} \ \lambda + \mu_j \geq \hat{a}_j x_j, \quad \lambda, \mu \geq 0.$$

The budgeted robust row is a finite LP constraint

For $(\bar{a} + \hat{a} \odot z)^\top x \le \beta$ in every budgeted scenario, introduce the inner dual variables:

$$\bar{a}^\top x + \Gamma\lambda + \sum_j \mu_j \le \beta, \qquad \lambda + \mu_j \ge \hat{a}_j x_j, \qquad \lambda, \mu \ge 0.$$

> Here the deviations for several coefficients compete inside one constraint. Increasing Γ beyond 1 can allow more coefficients to deviate together.

For the two-product example, $\bar{a} = (2, 1)$, $\hat{a} = (1, 1)$, and $\beta = 8$. At $\Gamma = 2$, the robust row is $3x_1 + 2x_2 \le 8$.

Construct the RHS dual with the correct signs

Let $U = \{b : Db \le d,\ b \ge \ell\} \neq \varnothing$. For row i, solve $\min_{b \in U} e_i^\top b$, where e_i has a 1 at position i.

Write both constraint groups as ≥: $-Db \ge -d$, $b \ge \ell$. Treat b as unrestricted; its lower bound is an explicit row.

$$\begin{aligned} \max \quad & -d^\top \lambda + \ell^\top \mu \\ \text{s.t.} \quad & -D^\top \lambda + \mu = e_i, \qquad \lambda, \mu \ge 0. \end{aligned}$$

The minus signs on D and d come from reversing the ≤ rows before forming a minimization dual.

A lower bound on every possible capacity

The robust row $a_i^\top x \le b_i$ for every $b \in U$ is equivalent to the existence of $\lambda_i, \mu_i \ge 0$ satisfying

$$-D^\top \lambda_i + \mu_i = e_i, \qquad a_i^\top x \le -d^\top \lambda_i + \ell^\top \mu_i.$$

Indeed, for every $b \in U$,

$$b_i = -\lambda_i^\top D b + \mu_i^\top b \ge -d^\top \lambda_i + \ell^\top \mu_i \ge a_i^\top x.$$

Conversely, strong duality supplies such variables if the nonempty inner LP has a finite minimum and the robust row holds.

Why different worst scenarios still constrain one fixed plan

$$\left[\forall b \in U : \; a_1^\top x \le b_1 \text{ and } a_2^\top x \le b_2\right]$$

$$\iff \quad \left[\forall b \in U : \; a_1^\top x \le b_1\right] \text{ and } \left[\forall b \in U : \; a_2^\top x \le b_2\right].$$

Each row must hold for all scenarios. The scenario producing row 1's minimum may differ from the scenario producing row 2's minimum.

> Coupling the right-hand sides matters only if it changes at least one of these separate minima, when all decisions are fixed before uncertainty is observed.

Verify the robust production optimum without a solver

The robust LP is $\max 40x_1 + 30x_2$ subject to $2x_1 + x_2 \le 80$, $x_1 + 2x_2 \le 60$, $x \ge 0$.

$$\frac{50}{3}(2x_1 + x_2) + \frac{20}{3}(x_1 + 2x_2) = 40x_1 + 30x_2 \le \frac{5200}{3}.$$

The plan $(100/3, 40/3)$ makes both inequalities active and attains the bound.

> For $t = \min\{1, \Gamma\}$, capacities are $(100 - 20t, 80 - 20t)$, quantities are $(40 - 20t/3, 20 - 20t/3)$, and profit is $2200 - 1400t/3$.

Portfolio protection under box uncertainty

Let $x_j \geq 0$ be the fraction invested in asset j, with $\sum_j x_j = 1$. Returns satisfy $\bar{r}_j - \hat{r}_j \leq r_j \leq \bar{r}_j + \hat{r}_j$.

$$\max_{x \geq 0,\, \mathbf{1}^\top x = 1} \; \min_{r \in U} r^\top x.$$

Because each $x_j \geq 0$, the smallest return chooses every $r_j = \bar{r}_j - \hat{r}_j$. Thus the model becomes

$$\max \; (\bar{r} - \hat{r})^\top x \quad \text{s.t. } \mathbf{1}^\top x = 1, \quad x \geq 0.$$

This simple box model need not diversify. Its constraints and uncertainty set must capture the intended investment restrictions.

A numerical portfolio comparison

Asset	Nominal return	Maximum drop	Worst return
A	8%	6%	2%
B	6%	1%	5%

With nonnegative fractions summing to 1 and no other restrictions, the nominal model puts everything in A; the robust box model puts everything in B.

> The guaranteed return increases from 2% to 5%, while the nominal return decreases from 8% to 6%. Neither model promises what will happen outside its specified scenarios.

Which uncertainty sets preserve a linear program?

Set	Geometry	Robust counterpart
Box	Coordinate bounds	LP
Budget / absolute-value bounds	Polyhedron after linearization	LP
Ellipsoid / Euclidean ball	Curved boundary in general	Usually a norm constraint, not an LP

$$\max_{\|a-\bar{a}\|_2 \le \rho} a^\top x = \bar{a}^\top x + \rho \|x\|_2.$$

An ellipsoid is not generally polyhedral. The Euclidean-norm formula is a contrast showing where the LP dualization argument changes.

Correlated delays: a continuous routing model

Two corridors each have two arcs. Corridor A takes 5+5 minutes nominally; B takes 6+6. A shared incident budget allows one unit of disruption.

$$z_A, z_B \geq 0, \quad z_A + z_B \leq 1.$$

Each A arc gains $4z_A$ minutes; each B arc gains $3z_B$. Let $t \in [0,1]$ be the fraction of flow sent on A, and $1 - t$ the fraction on B.

$$\min_{0 \leq t \leq 1} \Big[10t + 12(1-t) + \max_{z_A + z_B \leq 1} \left(8t z_A + 6(1-t) z_B \right) \Big].$$

Dualizing the adversary permits a split flow

The worst surcharge is $\max\{8t, 6(1-t)\}$. Introduce its upper bound q:

$$\begin{aligned} \min \quad & 12 - 2t + q \\ \text{s.t.} \quad & q \geq 8t, \quad q \geq 6(1-t), \quad 0 \leq t \leq 1. \end{aligned}$$

$$8t = 6(1-t) \Rightarrow t = 3/7, \qquad \text{value} = 102/7 \approx 14.57.$$

Sending everything on either single corridor gives worst-case time 18. The continuous model improves the guarantee by splitting flow. It does not select one path.

The general robust flow reformulation

Let x be a continuous feasible flow, $\bar{c}$ the nominal arc costs, and Δ the matrix mapping incidents $z \geq 0$ to added costs. Let $Bz \leq h$ describe incident budgets.

$$\min_{x \in P} \left[\bar{c}^{\top} x + \max_{z \geq 0,\, Bz \leq h} (\Delta^{\top} x)^{\top} z \right].$$

$$\min_{x \in P,\, \pi \geq 0} \quad \bar{c}^{\top} x + h^{\top} \pi \quad \text{s.t. } B^{\top} \pi \geq \Delta^{\top} x.$$

P contains flow conservation and continuous capacity bounds. The extra robust constraints need not preserve integral solutions. Binary single-path restrictions would create an integer-programming model.

Another price system: two-node generation dispatch

Node 2 needs 100 units of energy. Generator 1 costs 20 per unit, generator 2 costs 50; at most 60 units can move from node 1 to node 2.

$$\begin{aligned} \min \quad & 20g_1 + 50g_2 \\ \text{s.t.} \quad & g_1 - f = 0, \quad g_2 + f = 100, \quad 0 \le f \le 60, \quad g_1, g_2 \ge 0. \end{aligned}$$

Set $g_1 = f$, $g_2 = 100 - f$. Every extra transferred unit saves 30, so $(g_1, g_2, f) = (60, 40, 60)$, with cost 3200.

This transport-style model omits the voltage-angle equations of a physical power-flow model.

Derive and verify the dispatch dual

Use unrestricted p_1, p_2 for the two balance equalities and $-\eta \le 0$ for the upper bound $f \le 60$.

$$\begin{aligned} \max \quad & 100p_2 - 60\eta \\ \text{s.t.} \quad & p_1 \le 20, \quad p_2 \le 50, \quad -p_1 + p_2 - \eta \le 0, \quad \eta \ge 0. \end{aligned}$$

$$p_1 = 20, \quad p_2 = 50, \quad \eta = 30, \qquad 100(50) - 60(30) = 3200.$$

The optimal dual variables give the marginal demand costs at the nodes and the value of relaxing the transfer limit.

Capacity has value until the next change of basis

Replace 60 by $U \geq 0$. For $0 \leq U \leq 100$, use $f = U$, $g_1 = U$, and $g_2 = 100 - U$.

$$C(U) = 5000 - 30U \quad (0 \leq U \leq 100), \qquad C(U) = 2000 \quad (U \geq 100).$$

> One extra unit of transfer capacity saves 30 while the line is the limiting resource. Beyond 100, it adds no value in this model.

Expanding from 60 to 100 saves 1200 per modeled operating period. A cost of 25 per unit of added capacity gives a modeled net benefit $1200 - 25(40) = 200$.

A smaller booking model with an active demand limit

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 + 6x_3 \\ \text{s.t.} \quad & x_1 + x_3 \le 4, \quad x_2 + x_3 \le 4, \\ & 0 \le x_1, x_2 \le 4, \quad 0 \le x_3 \le 3. \end{aligned}$$

Products 1 and 2 are local bookings; product 3 uses both legs. A connection earns 6 versus 3+2 from the two locals it replaces.

$$x = (1, 1, 3), \qquad p = (3, 2), \quad u = (0, 0, 1), \qquad \text{primal value} = \text{dual value} = 23.$$

Separate capacity prices from demand prices

For the preceding two-leg model, keep connection demand equal to 3. With capacities C_1 and C_2, the same allocation pattern is

$$x_1 = C_1 - 3, \quad x_2 = C_2 - 3, \quad x_3 = 3, \qquad 3 \le C_1, C_2 \le 7.$$

$$V(C_1, C_2) = 3C_1 + 2C_2 + 3.$$

Instead keep capacities at 4 and let connecting demand be D. For $0 \le D \le 4$,

$$x = (4 - D, 4 - D, D), \qquad V(D) = 20 + D.$$

> The capacity prices are 3 and 2; the connection-demand price is 1. They answer different perturbation questions.

Competition for time slots also produces resource prices

Jobs A and B each need one slot; one peak and one off-peak slot are available.

Value	Peak	Off-peak
A	10	6
B	9	8

Let $x_{it} \geq 0$ allocate job i to slot t. Maximize the displayed values times x_{it}, subject to one unit per job and at most one unit per slot.

> A in peak and B in off-peak earns 18; the reversed assignment earns 15. Slot-capacity dual variables measure the opportunity cost of that competition.

The assignment lecture will develop the full primal–dual algorithm.

A local price is one input to an investment decision

Scenario	Hours	Value per hour	Annual value
High load	1000	30	30000
Medium load	2000	10	20000
Low load	5760	0	0

> One unit of capacity has gross annual value 50,000 under these scenarios. Multiplying a peak-hour price by all hours would overstate its value.

Keep units and validity ranges explicit. Reoptimize after substantial changes, and account for operational restrictions missing from the LP.

Different optimal dual solutions can give different prices

Degeneracy or redundant constraints can produce multiple optimal dual vectors. Each remains a valid certificate, but a single reported coordinate may not describe both directions of a perturbation.

> Check the proposed direction by changing the data and solving again. The directional change in value is more reliable than interpreting an arbitrary reported coordinate in isolation.

At a corner of the optimal-value function, one-sided marginal values can differ.

Study guide and next connection

Be able to: construct the booking dual, verify the allocation and certificate, explain displacement, and turn a worst-case linear problem into a finite robust LP.

Keep distinct: coefficient and RHS uncertainty; capacity and demand prices; a fixed plan and a plan that adapts; continuous split flow and a single path.

Read the complete derivations here before applying a formula to a new sign convention. Large-scale LP next uses reduced costs to select columns and separation to select constraints.