# Sensitivity Analysis: Ranges and Reoptimization

An optimal plan answers one set of data. Sensitivity analysis tells us what survives when those data change.

> **Our route:** update a familiar dictionary → find allowable ranges → choose a reoptimization method → interpret the whole value function.

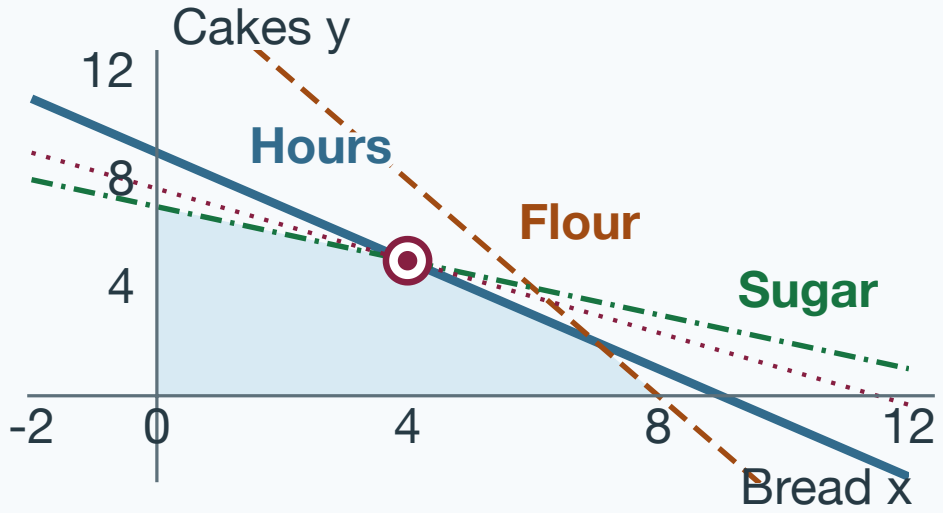
Bertsimas and Tsitsiklis, Chapter 5.

- A bakery example
- Right-hand sides
- Objective coefficients
- Solver reports
- Global value functions

# One bakery, two products, three resources

Let $x$ be bread production and $y$ cake production. Quantities are continuous, measured in batches.

$$
\begin{aligned}
\max \quad & z = 2x + 3y \\
\text{s.t.} \quad & x + y \le 9 && \text{(hours)}, \\
& 2x + y \le 16 && \text{(flour)}, \\
& x + 2y \le 14 && \text{(sugar)}, \\
& x, y \ge 0.
\end{aligned}
$$




○ Original basis · ● Optimum
Shading: feasible plans · Dotted: profit line

The optimum is $(x, y) = (4, 5)$, with profit $23$.

Hours and sugar are fully used; flour has $3$ units left.

# Start from an optimal dictionary we can check

Add nonnegative slacks: $x + y + s_1 = 9$, $2x + y + s_2 = 16$, $x + 2y + s_3 = 14$.

$$
\begin{aligned}
x &= 4 - 2s_1 + s_3, \\
y &= 5 + s_1 - s_3, \\
s_2 &= 3 + 3s_1 - s_3, \\
z &= 23 - s_1 - s_3.
\end{aligned}
$$

> Basic variables: $x, y, s_2$. Nonbasic variables: $s_1, s_3$. Setting the latter to zero gives $(4, 5, 3)$.

This is **maximization**: the nonbasic objective coefficients $-1, -1$ certify optimality. Increasing a nonnegative slack cannot improve profit.

# After any change, check two separate things

| Question | For this maximizing dictionary |
| --- | --- |
| **Primal feasibility** | Are all basic values nonnegative? |
| **Dual feasibility** | Are all nonbasic objective coefficients $\leq 0$? |

> If both checks pass, the current basis is still optimal.

A change in a right-hand side affects the basic values. A change in a profit coefficient affects the objective row.

We will do the arithmetic first, then summarize it in matrix form.

# Change the flour limit

Maximize $2x + 3y$; $x + y \le 9,\ 2x + y \le 16 + \delta,\ x + 2y \le 14,\ x, y \ge 0$.

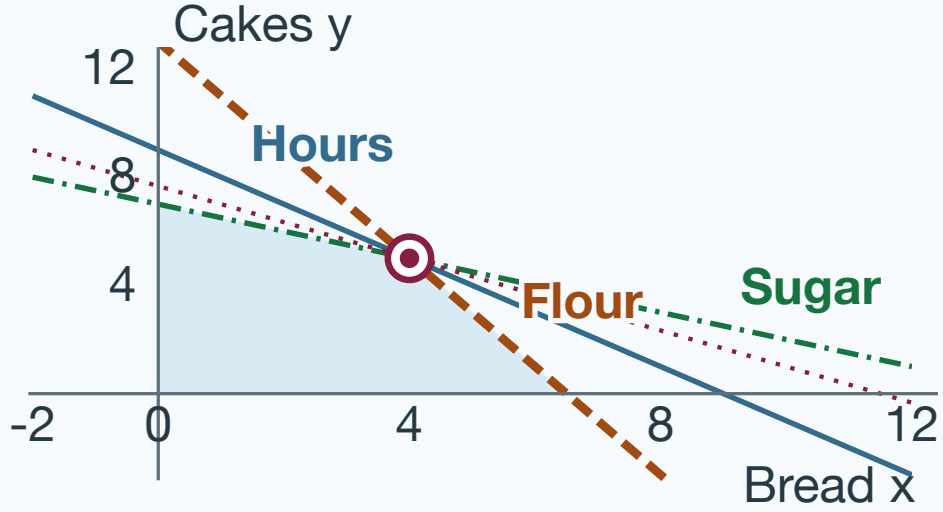
## Original dictionary

$$x = 4 - 2s_1 + s_3,$$
$$y = 5 + s_1 - s_3,$$
$$s_2 = 3 + 3s_1 - s_3,$$
$$z = 23 - s_1 - s_3.$$

## After the flour change

$$x = 4 - 2s_1 + s_3,$$
$$y = 5 + s_1 - s_3,$$
$$s_2 = 3 + \delta + 3s_1 - s_3,$$
$$z = 23 - s_1 - s_3.$$

Set $s_1 = s_3 = 0$. Feasible exactly when $3 + \delta \ge 0$: $\delta \ge -3$. Profit stays 23.




○ Original basis · ● Optimum
Shading: feasible plans · Dotted: profit line

**Shown:** $\delta = -3$. Flour limit: 13. Original basis: 4 bread, 5 cakes, 0 unused flour (feasible). Profit 23; optimum 23.

# Change the hours limit

Maximize $2x + 3y$; $x + y \le 9 + \delta,\ 2x + y \le 16,\ x + 2y \le 14,\ x, y \ge 0$.

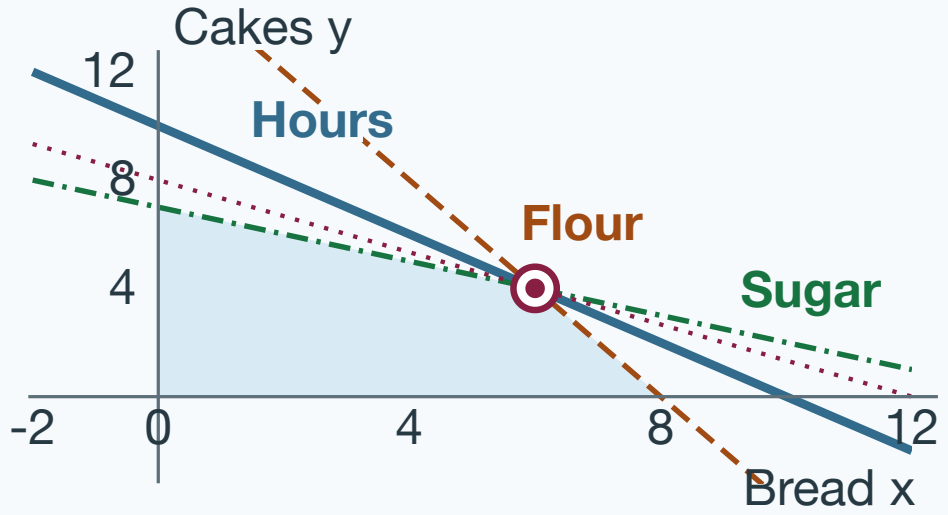
## Original dictionary

$$x = 4 - 2s_1 + s_3,$$
$$y = 5 + s_1 - s_3,$$
$$s_2 = 3 + 3s_1 - s_3,$$
$$z = 23 - s_1 - s_3.$$

## After the hours change

$$x = 4 + 2\delta - 2s_1 + s_3,$$
$$y = 5 - \delta + s_1 - s_3,$$
$$s_2 = 3 - 3\delta + 3s_1 - s_3,$$
$$z = 23 + \delta - s_1 - s_3.$$

Require $4 + 2\delta,\ 5 - \delta,\ 3 - 3\delta \ge 0$. Thus $-2 \le \delta \le 1$; profit is $23 + \delta$.




○ Original basis · ● Optimum
Shading: feasible plans · Dotted: profit line

**Shown:** $\delta = 1$. Hours limit: 10. Original basis: 6 bread, 4 cakes, 0 unused flour (feasible). Profit 24; optimum 24.

# Practice: change the sugar limit

Maximize $2x + 3y$; $x + y \le 9,\ 2x + y \le 16,\ x + 2y \le 14 + \delta,\ x, y \ge 0$.

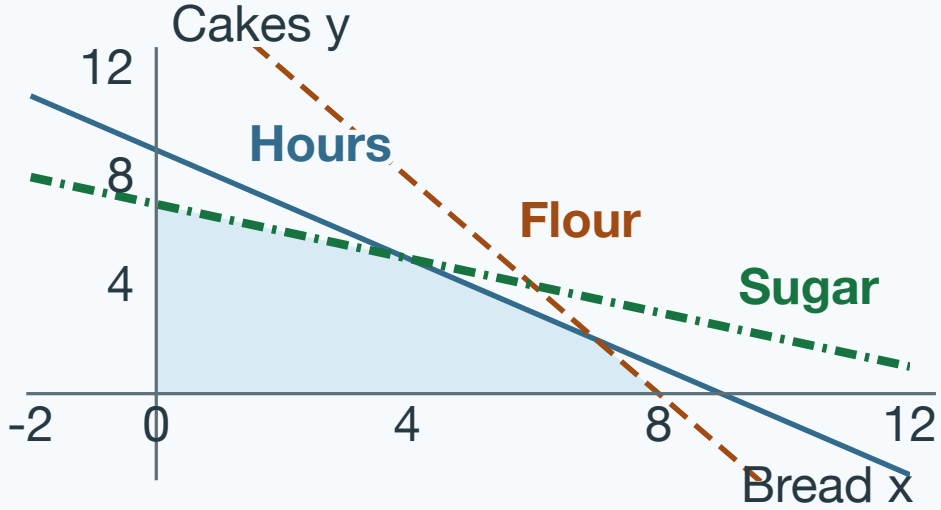
## Original dictionary

$$x = 4 - 2s_1 + s_3,$$
$$y = 5 + s_1 - s_3,$$
$$s_2 = 3 + 3s_1 - s_3,$$
$$z = 23 - s_1 - s_3.$$

Keep $s_1 = s_3 = 0$. Find the **updated dictionary**, the allowable interval for $\delta$, and the profit.

Use $x + y = 9 - s_1$, $x + 2y = 14 + \delta - s_3$, then $s_2 = 16 - 2x - y$.

Take two minutes. Include unused flour in the feasibility test.




**Shown:** $\delta = 0$. Sugar limit: 14. Find the updated dictionary and its allowable interval.

# Sugar can decrease by 3 or increase by 4

Maximize $2x + 3y$; $x + y \le 9,\ \ 2x + y \le 16,\ \ x + 2y \le 14 + \delta,\ \ x, y \ge 0.$

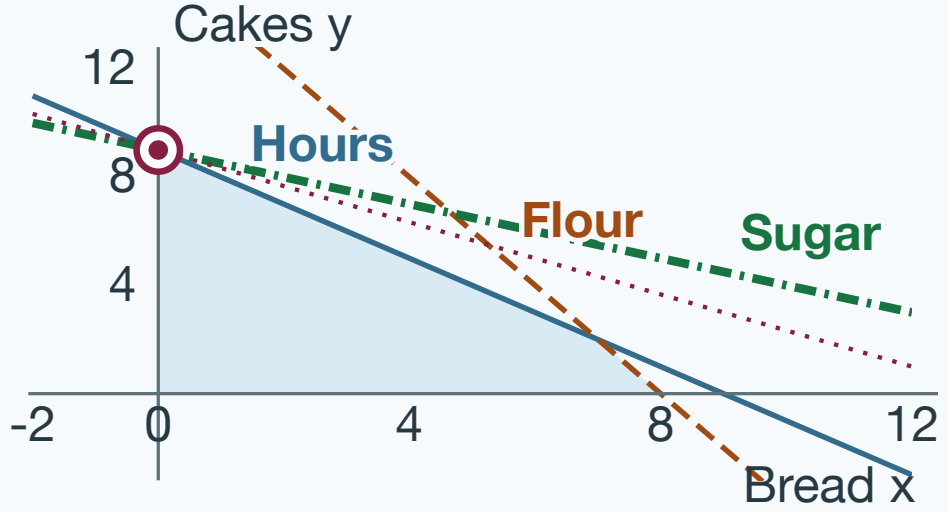
## Original dictionary

$$x = 4 - 2s_1 + s_3,$$
$$y = 5 + s_1 - s_3,$$
$$s_2 = 3 + 3s_1 - s_3,$$
$$z = 23 - s_1 - s_3.$$

## After the sugar change

$$x = 4 - \delta - 2s_1 + s_3,$$
$$y = 5 + \delta + s_1 - s_3,$$
$$s_2 = 3 + \delta + 3s_1 - s_3,$$
$$z = 23 + \delta - s_1 - s_3.$$

Feasible exactly for $-3 \le \delta \le 4$: flour slack reaches zero at –3; bread at 4. Profit: $23 + \delta$.




○ Original basis · ● Optimum
Shading: feasible plans · Dotted: profit line

**Shown:** $\delta = 4$. Sugar limit: 18. Original basis: 0 bread, 9 cakes, 7 unused flour (feasible). Profit 27; optimum 27.

# Recall the bakery and its optimal dictionary

$x$: bread batches; $y$: cake batches. Maximize $z = 2x + 3y$.

Original limits: $x + y \le 9$ (hours), $2x + y \le 16$ (flour), $x + 2y \le 14$ (sugar); $x, y \ge 0$.

Introduce nonnegative slacks $s_1, s_2, s_3$, the unused hours, flour, and sugar:

$$x + y + s_1 = 9, \qquad 2x + y + s_2 = 16, \qquad x + 2y + s_3 = 14.$$

## Original optimal dictionary

$$\begin{aligned} x &= 4 - 2s_1 + s_3, \\ y &= 5 + s_1 - s_3, \\ s_2 &= 3 + 3s_1 - s_3, \\ z &= 23 - s_1 - s_3. \end{aligned}$$

Basic: $x, y, s_2$. Nonbasic: $s_1, s_3$. Set $s_1 = s_3 = 0$: $(x, y, s_2) = (4, 5, 3)$, profit $23$. The objective coefficients $-1, -1$ are nonpositive, as required for maximization.

# Update the hours dictionary, then set δ = 2

Maximize $2x + 3y$; $x + y \le 9 + \delta,\ 2x + y \le 16,\ x + 2y \le 14,\ x, y \ge 0.$

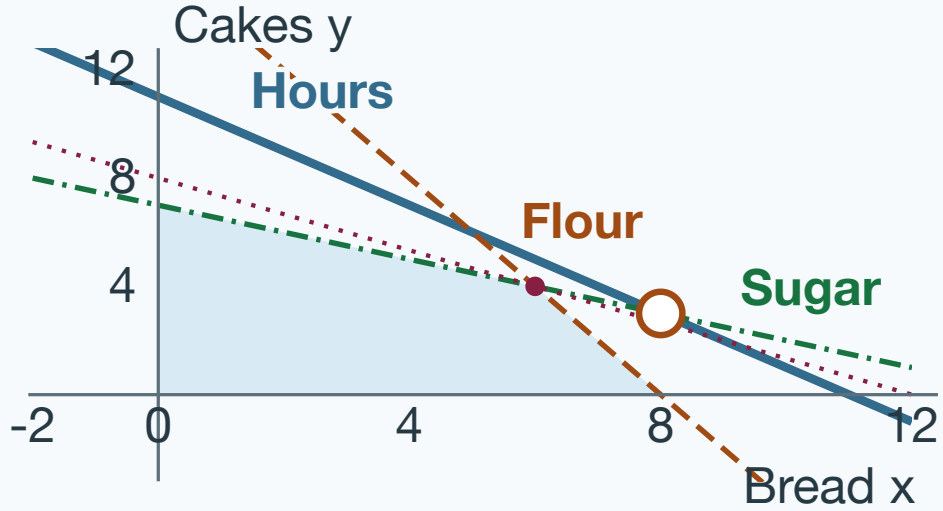
Old hours slack: $s_1^{\text{old}} = 9 - x - y.$

New hours slack: $s_1 = 9 + \delta - x - y.$

Substitute $s_1^{\text{old}} = s_1 - \delta$ in every original row:

| Variable | With $\delta$ | $\delta = 2$ |
|---|---|---|
| $x$ | $4 + 2\delta - 2s_1 + s_3$ | $8 - 2s_1 + s_3$ |
| $y$ | $5 - \delta + s_1 - s_3$ | $3 + s_1 - s_3$ |
| $s_2$ | $3 - 3\delta + 3s_1 - s_3$ | $-3 + 3s_1 - s_3$ |
| $z$ | $23 + \delta - s_1 - s_3$ | $25 - s_1 - s_3$ |

At $\delta = 2$, $s_1 = s_3 = 0$ gives $s_2 = -3$. **Leave $s_2$; enter $s_1$**, the only variable with a positive coefficient in that row. Reduced costs stay $-1, -1$: dual feasible.




○ Original basis · ● Optimum
Shading: feasible plans · Dotted: profit line

**Shown:** $\delta = 2$. Hours limit: 11. Original basis: 8 bread, 3 cakes, -3 unused flour (infeasible). Profit 25; optimum 24.

# One dual-simplex pivot restores feasibility

Maximize $2x + 3y$, with $x + y \le 11$, $2x + y \le 16$, $x + 2y \le 14$, $x, y \ge 0$.

From $s_2 = -3 + 3s_1 - s_3$, solve $s_1 = 1 + \frac{1}{3}s_2 + \frac{1}{3}s_3$ and substitute in every row.

| Before pivot | After pivot |
|---|---|
| $x = 8 - 2s_1 + s_3$ | $x = 6 - \frac{2}{3}s_2 + \frac{1}{3}s_3$ |
| $y = 3 + s_1 - s_3$ | $y = 4 + \frac{1}{3}s_2 - \frac{2}{3}s_3$ |
| $s_2 = -3 + 3s_1 - s_3$ | $s_1 = 1 + \frac{1}{3}s_2 + \frac{1}{3}s_3$ |
| $z = 25 - s_1 - s_3$ | $z = 24 - \frac{1}{3}s_2 - \frac{4}{3}s_3$ |

Set $s_2 = s_3 = 0$: $(x, y, s_1) = (6, 4, 1)$, profit **24**. All basic values are nonnegative and all nonbasic objective coefficients are nonpositive.

The old value 25 was an upper bound; it was not an attainable profit.

# Two allowable changes may fail together

**Bakery:** maximize $z = 2x + 3y$, with $x + y \leq 9$, $2x + y \leq 16$, $x + 2y \leq 14$, $x, y \geq 0$.

**Independent variation:** return to basis $(x, y, s_2)$, with $s_1 = s_3 = 0$. Change hours by $\delta_1$ and flour by $\delta_2$; keep sugar at 14.

From $x + y = 9 + \delta_1$, $x + 2y = 14$, and $s_2 = 16 + \delta_2 - 2x - y$:

Thus $x = 4 + 2\delta_1, \quad y = 5 - \delta_1, \quad s_2 = 3 - 3\delta_1 + \delta_2$.

| Change from the original bakery | $x$ | $y$ | Flour slack $s_2$ |
|---|---|---|---|
| **None** | 4 | 5 | 3 |
| **Hours $9 \to 10$ only** | 6 | 4 | 0 |
| **Flour $16 \to 13$ only** | 4 | 5 | 0 |
| **Both changes together** | 6 | 4 | $-3$ |

With 10 hours, the plan $(6, 4)$ uses $2(6) + 4 = 16$ units of flour. Only 13 are available after both changes: **this basis is infeasible**. The changed LP may still be feasible.

# The 100% rule gives a sufficient check

**Bakery:** maximize $z = 2x + 3y$, with $x + y \le 9$, $2x + y \le 16$, $x + 2y \le 14$, $x, y \ge 0$.

Original basis: $(x, y, s_2)$. Divide each change's magnitude by its allowance in the table; add.

| Resource change | One-at-a-time allowance | Fraction used |
|---|---|---|
| **Hours** $9 \to 9.5$ | Increase by 1: $9 \to 10$ | $0.5/1 = 50\%$ |
| **Flour** $16 \to 15$ | Decrease by 3: $16 \to 13$ | $1/3 \approx 33.3\%$ |

Sum of fractions: $R = \frac{0.5}{1} + \frac{1}{3} = \frac{5}{6} < 1$.

$R \le 1$ guarantees a feasible basis. For hours $+0.5$ and flour $-1$:

$$x = 5, \quad y = 4.5, \quad s_2 = 3 - 3(0.5) - 1 = 0.5.$$

The objective coefficients remain $-1, -1$, so this basis is also **optimal**; profit is $23.5$.

Percentages refer to the **allowable changes**, not the original resource totals. If $R > 1$, check the basic values directly; the rule gives no conclusion.

Why the rule works · A feasible example above 100%

# Change bread profit using the original dictionary

**Start again from the original bakery:** hours 9, flour 16, sugar 14; maximize $2x + 3y$.

Change bread profit from $2$ to $2 + \theta$: $z_{\text{new}} = z_{\text{old}} + \theta x$. Substitute the original row for $x$.

| Variable | Original dictionary | Updated dictionary |
|---|---|---|
| $x$ | $4 - 2s_1 + s_3$ | $4 - 2s_1 + s_3$ |
| $y$ | $5 + s_1 - s_3$ | $5 + s_1 - s_3$ |
| $s_2$ | $3 + 3s_1 - s_3$ | $3 + 3s_1 - s_3$ |
| $z$ | $23 - s_1 - s_3$ | $23 + 4\theta + (-1 - 2\theta)s_1 + (-1 + \theta)s_3$ |

The three constraint rows are unchanged. For maximization, require $-1 - 2\theta \leq 0$ and $-1 + \theta \leq 0$: $-\frac{1}{2} \leq \theta \leq 1$, or $1.5 \leq c_x \leq 3$.

Within this interval, quantities stay $(4, 5)$ and profit becomes $23 + 4\theta$.

# Selling unused hours changes one objective coefficient

**Start again from the original bakery:** hours 9, flour 16, sugar 14; maximize $2x + 3y$.

Suppose unused hours $s_1$ can be sold for $\kappa$ per unit. Then $z_{\text{new}} = z_{\text{old}} + \kappa s_1$.

| Variable | Original dictionary | Updated dictionary |
|---|---|---|
| $x$ | $4 - 2s_1 + s_3$ | $4 - 2s_1 + s_3$ |
| $y$ | $5 + s_1 - s_3$ | $5 + s_1 - s_3$ |
| $s_2$ | $3 + 3s_1 - s_3$ | $3 + 3s_1 - s_3$ |
| $z$ | $23 - s_1 - s_3$ | $23 + (\kappa - 1)s_1 - s_3$ |

The original basis remains optimal exactly when $\kappa \leq 1$. Its profit stays 23 because $s_1 = 0$. For $\kappa > 1$, **primal simplex** can improve the still-feasible plan.

# Change both profits using the same original rows

**Start again from the original bakery:** hours 9, flour 16, sugar 14; maximize $2x + 3y$.

Set profits to $2 + \theta_1$ and $3 + \theta_2$. Use $z_{\text{new}} = z_{\text{old}} + \theta_1 x + \theta_2 y$.

| Variable | Original dictionary | Updated dictionary |
|---|---|---|
| $x$ | $4 - 2s_1 + s_3$ | $4 - 2s_1 + s_3$ |
| $y$ | $5 + s_1 - s_3$ | $5 + s_1 - s_3$ |
| $s_2$ | $3 + 3s_1 - s_3$ | $3 + 3s_1 - s_3$ |
| $z$ | $23 - s_1 - s_3$ | $23 + 4\theta_1 + 5\theta_2 + (-1 - 2\theta_1 + \theta_2)s_1 + (-1 + \theta_1 - \theta_2)s_3$ |

Both objective coefficients must be nonpositive: $\theta_1 - 1 \le \theta_2 \le 1 + 2\theta_1$. If only cake profit changes, $-1 \le \theta_2 \le 1$, so $2 \le c_y \le 4$.

# Write the bakery equations in matrix form

**Bakery:** maximize $z = 2x + 3y$, with $x + y \le 9$, $2x + y \le 16$, $x + 2y \le 14$, $x, y \ge 0$.

Slacks $s_1, s_2, s_3 \ge 0$ are unused hours, flour, sugar. In $A\mathbf{x} = b$, order $\mathbf{x} = (x, y, s_1, s_2, s_3)^\top$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 9 \\ 16 \\ 14 \end{pmatrix}.$$

Use the basis already found by simplex: $(x, y, s_2)$. Let $A_{\mathcal{N}}$ contain the remaining columns, for $(s_1, s_3)$.

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}, \qquad A_{\mathcal{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$B(x, y, s_2)^\top + A_{\mathcal{N}}(s_1, s_3)^\top = b.$$

Solve for $(x, y, s_2)$: subtract the nonbasic terms, then multiply by $B^{-1}$.

# Compute the same dictionary using the inverse

**Bakery:** maximize $z = 2x + 3y$, with $x + y \le 9$, $2x + y \le 16$, $x + 2y \le 14$, $x, y \ge 0$.

$$(x, y, s_2)^\top = B^{-1}b - B^{-1}A_{\mathcal{N}}(s_1, s_3)^\top.$$

$$B^{-1}b = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 16 \\ 14 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}.$$

$$B^{-1}A_{\mathcal{N}} = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ -3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \\ -3 & 1 \end{pmatrix}.$$

$$x = 4 - 2s_1 + s_3, \qquad y = 5 + s_1 - s_3, \qquad s_2 = 3 + 3s_1 - s_3.$$

For the objective, substitute these expressions for $x, y$:

$$z = 2(4 - 2s_1 + s_3) + 3(5 + s_1 - s_3) = 23 - s_1 - s_3.$$

# A basis determines dual equalities

**Bakery:** maximize $z = 2x + 3y$, with $x + y \leq 9$, $2x + y \leq 16$, $x + 2y \leq 14$, $x, y \geq 0$.

Use basis $(x, y, s_2)$, where $s_2$ is unused flour. To write $z$ using only $s_1, s_3$, multiply the hours, flour and sugar equalities by $p_1, p_2, p_3$, then add. Match the basic-variable coefficients in $z = 2x + 3y$:

| Basic variable | Objective coefficient | Coefficient-matching equation |
|---|---|---|
| $x$ | 2 | $p_1 + 2p_2 + p_3 = 2$ |
| $y$ | 3 | $p_1 + p_2 + 2p_3 = 3$ |
| $s_2$ | 0 | $p_2 = 0$ |

These equations are $B^\top p = c_{\mathcal{B}}$; $c_{\mathcal{B}} = (2, 3, 0)^\top$ lists the basic objective coefficients.

$$\begin{aligned} p &= (B^\top)^{-1} c_{\mathcal{B}} = (B^{-1})^\top c_{\mathcal{B}} \\ &= (1, 0, 1)^\top. \end{aligned}$$

**Every basis** determines a unique $p$ making its basic-variable dual constraints equalities. This $p$ is a **dual candidate**: we must still check the other dual constraints.

# Check dual feasibility to establish optimality

**Bakery:** maximize $z = 2x + 3y$, with $x + y \le 9$, $2x + y \le 16$, $x + 2y \le 14$, $x, y \ge 0$.

The dual variables $p_1, p_2, p_3$ correspond to hours, flour, sugar:

$$\begin{aligned} \min \quad & 9p_1 + 16p_2 + 14p_3 \\ \text{s.t.} \quad & p_1 + 2p_2 + p_3 \ge 2, \quad p_1 + p_2 + 2p_3 \ge 3, \quad p \ge 0. \end{aligned}$$

Check $p = (1, 0, 1)^\top$: $1 + 2(0) + 1 = 2$, $1 + 0 + 2(1) = 3$, and $p \ge 0$. All dual constraints hold. Adding the hours and sugar equations gives

$$2x + 3y + s_1 + s_3 = 23 \quad \Longrightarrow \quad z = 23 - s_1 - s_3.$$

The primal point $(x, y) = (4, 5)$ is feasible. Its value $2(4) + 3(5) = 23$ equals the dual value $9(1) + 16(0) + 14(1) = 23$. By **weak duality, both solutions are optimal**.

**A different basis:** the initial slack basis $(s_1, s_2, s_3)$ gives $p = (0, 0, 0)$. It makes the three slack-variable dual constraints equalities, but violates $0 \ge 2$ and $0 \ge 3$. A basis need not give a dual-feasible vector.

# Derive the objective change when the RHS changes

**Bakery:** maximize $z = 2x + 3y$, with $x + y \le 9$, $2x + y \le 16$, $x + 2y \le 14$, $x, y \ge 0$.

Change only the resource limits by $\Delta b = (\delta_1, \delta_2, \delta_3)^\top$ (hours, flour, sugar). Keep basis $(x, y, s_2)$ and set $s_1 = s_3 = 0$.

$$x_{\mathcal{B}} = (x, y, s_2)^\top, \qquad x_{\mathcal{B}}^{\text{old}} = B^{-1} b, \qquad x_{\mathcal{B}}^{\text{new}} = B^{-1}(b + \Delta b).$$

$$\begin{aligned} z^{\text{new}} &= c_{\mathcal{B}}^\top x_{\mathcal{B}}^{\text{new}} = c_{\mathcal{B}}^\top B^{-1}(b + \Delta b) \\ &= c_{\mathcal{B}}^\top B^{-1} b + c_{\mathcal{B}}^\top B^{-1} \Delta b \\ &= z^{\text{old}} + p^\top \Delta b = 23 + \delta_1 + \delta_3. \end{aligned}$$

Here $p^\top = c_{\mathcal{B}}^\top B^{-1}$. Adding hours and sugar also gives $z + s_1 + s_3 = 23 + \delta_1 + \delta_3$.

**This is the new optimum if the updated basic values are nonnegative:**

$$x = 4 + 2\delta_1 - \delta_3 \ge 0, \quad y = 5 - \delta_1 + \delta_3 \ge 0, \quad s_2 = 3 - 3\delta_1 + \delta_2 + \delta_3 \ge 0.$$

Reduced costs stay $-1, -1$. A negative basic value requires reoptimization.

# Add a variable: solve for the existing basic variables

Let $t \geq 0$ count pastry batches. **Maximize** $z = 2x + 3y + 4t$, with nonnegative slacks for unused resources:

$$\begin{aligned} H\ (\text{hours}): &\quad x + y + s_1 + t = 9, \\ F\ (\text{flour}): &\quad 2x + y + s_2 + 2t = 16, \\ S\ (\text{sugar}): &\quad x + 2y + s_3 + t = 14, \qquad x, y, s_1, s_2, s_3 \geq 0. \end{aligned}$$

Keep $(x, y, s_2)$ basic. **Solve for these variables** in terms of $(s_1, s_3, t)$, using elimination:

| Isolate | Combine original rows | Constraint dictionary |
|---|---|---|
| $x$ | $2H - S$ | $x = 4 - 2s_1 + s_3 - t$ |
| $y$ | $S - H$ | $y = 5 + s_1 - s_3$ |
| $s_2$ | $F - 3H + S$ | $s_2 = 3 + 3s_1 - s_3$ |

**No pivot yet:** $s_1 = s_3 = t = 0$ gives the feasible plan $(x, y, t) = (4, 5, 0)$. Next, connect these row operations to $B^{-1}$ and substitute into the objective.

# Compute the new dictionary column

**With pastry $t$:** maximize $z = 2x + 3y + 4t$.

$x + y + t \le 9$ (hours), $2x + y + 2t \le 16$ (flour), $x + 2y + t \le 14$ (sugar); $x, y, t \ge 0$.

Keep $x_{\mathcal{B}} = (x, y, s_2)^\top$, $x_{\mathcal{N}} = (s_1, s_3)^\top$. The columns of $A_{\mathcal{N}}$ belong to $s_1, s_3$; $A_t = (1, 2, 1)^\top$.

$$Bx_{\mathcal{B}} + A_{\mathcal{N}}x_{\mathcal{N}} + A_t t = b,$$

$$x_{\mathcal{B}} = \underbrace{B^{-1}b - B^{-1}A_{\mathcal{N}}x_{\mathcal{N}}}_{\text{old constraint dictionary}} - B^{-1}A_t t.$$

Rows of $B^{-1}$ combine the original hours $H$, flour $F$, sugar $S$ rows as $2H - S$, $S - H$, $F - 3H + S$:

$$-B^{-1}A_t = -\begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ -3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

The coefficient of $t$ is $-1$ in the $x$ equation and $0$ in the $y, s_2$ equations. This gives exactly the same constraint dictionary as elimination.

# Compute the new objective coefficient

**With pastry $t$:** maximize $z = 2x + 3y + 4t$.

$x + y + t \le 9$ (hours), $2x + y + 2t \le 16$ (flour), $x + 2y + t \le 14$ (sugar); $x, y, t \ge 0$.

The basis is still $(x, y, s_2)$. Substitute its equations into $z = 2x + 3y + 4t$:

$$\begin{aligned} x &= 4 - 2s_1 + s_3 - t, \\ y &= 5 + s_1 - s_3, \\ s_2 &= 3 + 3s_1 - s_3. \end{aligned}$$

$$\begin{aligned} z &= 2(4 - 2s_1 + s_3 - t) + 3(5 + s_1 - s_3) + 4t \\ &= 23 - s_1 - s_3 + 2t. \end{aligned}$$

Check with $p = (1, 0, 1)^\top$ and $A_t = (1, 2, 1)^\top$: the reduced-cost formula agrees.

$$\bar{c}_t = 4 - p^\top A_t = 4 - (1 + 0 + 1) = 2.$$

**Maximize:** increasing $t$ improves the objective since $\bar{c}_t = 2 > 0$. The **ratio test** determines how far we can increase it. Complete two-pivot solution

# Extra pastry hours: when is the old plan optimal?

**With pastry $t$:** maximize $z = 2x + 3y + 4t$.

$x + y + (1 + \delta)t \le 9$ (hours), $2x + y + 2t \le 16$ (flour), $x + 2y + t \le 14$ (sugar); $x, y, t \ge 0$.

**Before any pivot:** keep basis $(x, y, s_2)$. Only pastry hours rise, by $\delta \ge 0$. Add the hours and sugar equalities, including their slacks:

$$2x + 3y + (2 + \delta)t + s_1 + s_3 = 9 + 14 = 23.$$

$$z = 2x + 3y + 4t = 23 - s_1 - s_3 + (2 - \delta)t.$$

At $t = 0$, the changed coefficient multiplies zero: $(4, 5, 0)$ stays feasible. For **maximization**, require all nonbasic objective coefficients to be nonpositive:

$$-1 \le 0, \quad -1 \le 0, \quad 2 - \delta \le 0 \quad \Longleftrightarrow \quad \delta \ge 2.$$

Thus the plan $(4, 5, 0)$ is optimal for $\delta \ge 2$. For $0 \le \delta < 2$, continue primal simplex with $t$ entering.

# Derive the reduced-cost change

Change the coefficient of a **nonbasic** variable $x_j$ in constraint $i$ by $\delta$. Keep all other data fixed.

The basis $B$, its costs $c_{\mathcal{B}}$, and $p = (B^{-1})^\top c_{\mathcal{B}}$ stay fixed. Let $A_j^{\text{new}}$ denote the changed column.

Recall $\bar{c}_j = c_j - p^\top A_j$. Subtract the old reduced cost from the new one:

$$\begin{aligned} \bar{c}_j^{\text{new}} - \bar{c}_j &= (c_j - p^\top A_j^{\text{new}}) - (c_j - p^\top A_j) \\ &= -p^\top (A_j^{\text{new}} - A_j) = -p_i \delta. \end{aligned}$$

Only entry $i$ changes, so the change in the dot product is $p_i \delta$.

$$\boxed{\bar{c}_j^{\text{new}} = \bar{c}_j - \delta p_i.}$$

# Use the objective sense to decide: stop or pivot

For a **feasible dictionary**, nonbasic variables start at zero and must remain nonnegative. Write the objective as

$$z = z_0 + \sum_{j \in \mathcal{N}} \bar{c}_j x_j, \qquad x_j \geq 0.$$

| Objective | Stop when every nonbasic coefficient satisfies | Entering candidates |
|---|---|---|
| **Maximize** | $\bar{c}_j \leq 0$ | $\bar{c}_j > 0$ |
| **Minimize** | $\bar{c}_j \geq 0$ | $\bar{c}_j < 0$ |

A positive coefficient increases the objective; a negative coefficient decreases it. The ratio test determines an admissible step for an entering candidate.

> Change only the coefficient of a **nonbasic** variable $x_j$ in row $i$ by $\delta$. Since $x_j = 0$, the basic values stay feasible. Test $\bar{c}_j^{\text{new}} = \bar{c}_j - \delta p_i$; the other reduced costs are unchanged.

If a basic column changes: recompute the basis and reduced costs

# Start again with two products: add a bread limit

**Bakery:** maximize $z = 2x + 3y$, with $x + y \leq 9$, $2x + y \leq 16$, $x + 2y \leq 14$, $x, y \geq 0$.

## Original optimal dictionary

$$x = 4 - 2s_1 + s_3, \qquad y = 5 + s_1 - s_3,$$
$$s_2 = 3 + 3s_1 - s_3, \qquad z = 23 - s_1 - s_3.$$

**Separate variation: no pastry or recipe change.** Add $x \leq 3$. The original plan has $x = 4$, so it violates the new constraint.

$$x + s_4 = 3, \quad s_4 \geq 0; \qquad s_4 = 3 - (4 - 2s_1 + s_3) = -1 + 2s_1 - s_3.$$

Use basis $(x, y, s_2, s_4)$, with $s_1 = s_3 = 0$. The new basic value $s_4 = -1$ is negative, while reduced costs remain $-1, -1$. **Use dual simplex.**

If an added constraint is satisfied by the old optimum, that solution remains optimal.

## Complete repair pivot

# The failed check tells us how to continue

| Basic values feasible? | Reduced costs have correct signs? | Next step |
|---|---|---|
| **Yes** | Yes | Current basis remains optimal. |
| **Yes** | No | Primal simplex. |
| **No** | Yes | Dual simplex. |
| **No** | No | Repair feasibility, then optimize; Phase I is available. |

> Outside a sensitivity range, reoptimize. A failed basis test alone establishes neither infeasibility nor unboundedness.

# Bakery summary: ranges for the original optimal basis

For $\max z = 2x + 3y$, the **original optimal production quantities** are $x = 4$ bread batches and $y = 5$ cake batches. The tables show the original resource limits and one-at-a-time ranges:

| Resource | RHS | Slack | Dual variable | Absolute RHS interval |
|---|---|---|---|---|
| **Hours** | 9 | 0 | 1 | $[7, 10]$ |
| **Flour** | 16 | 3 | 0 | $[13, \infty)$ |
| **Sugar** | 14 | 0 | 1 | $[11, 18]$ |

| Product | Quantity (batches) | Profit coefficient | Reduced cost | Absolute profit interval |
|---|---|---|---|---|
| **Bread** | 4 | 2 | 0 | $[1.5, 3]$ |
| **Cake** | 5 | 3 | 0 | $[2, 4]$ |

Intervals give parameter values, not increments; change only one value at a time.

**Local bakery calculations end here.** Next: verify this report in software, then study changes across several optimal bases.

# Software verification: solve the original bakery

```
import gurobipy as gp
from gurobipy import GRB
m = gp.Model("bakery")
x = m.addVar(lb=0, name="bread")
y = m.addVar(lb=0, name="cake")
m.setObjective(2*x + 3*y, GRB.MAXIMIZE)
m.addConstr(x + y <= 9, name="hours")
m.addConstr(2*x + y <= 16, name="flour")
m.addConstr(x + 2*y <= 14, name="sugar")
m.Params.Method = 1  # dual simplex: obtain a basis
m.optimize()
```

The solution is continuous; these are batches, not integer counts.

# Software verification: read the sensitivity report

```
if m.Status == GRB.OPTIMAL:
    for row in m.getConstrs():
        print(row.ConstrName, row.Slack, row.Pi,
              row.SARHSLow, row.SARHSUp)
    for var in m.getVars():
        print(var.VarName, var.X, var.RC,
              var.SAObjLow, var.SAObjUp)
```

`Pi`: dual variable; RC: reduced cost. `SARHS…` and `SAObj…`: RHS and objective-coefficient limits. Sensitivity ranges require a basic solution.

At degeneracy, another optimal basis may report different valid ranges. A zero dual variable does not imply that the constraint is inactive.

Gurobi attribute reference

# From one optimal basis to the whole value function

**Three ideas to follow:** within an optimal basis's allowable range, the optimal value changes linearly. At a boundary, another basis may take over. A dual variable gives the slope on its applicable linear piece.

Now write the bakery as **minimization** of $f = -z = -2x - 3y$. The feasible set is unchanged.

The **optimal-value function** $F(b)$ records the minimum cost for each RHS $b$, wherever the optimum is finite. Strong duality gives

$$\begin{aligned} F(b) &= \min\{c^\top x : Ax = b,\ x \geq 0\} \\ &= \max_{A^\top p \leq c} p^\top b. \end{aligned}$$

Each feasible dual vector supplies a linear lower bound. Taking the best bound produces a convex, piecewise-linear graph.

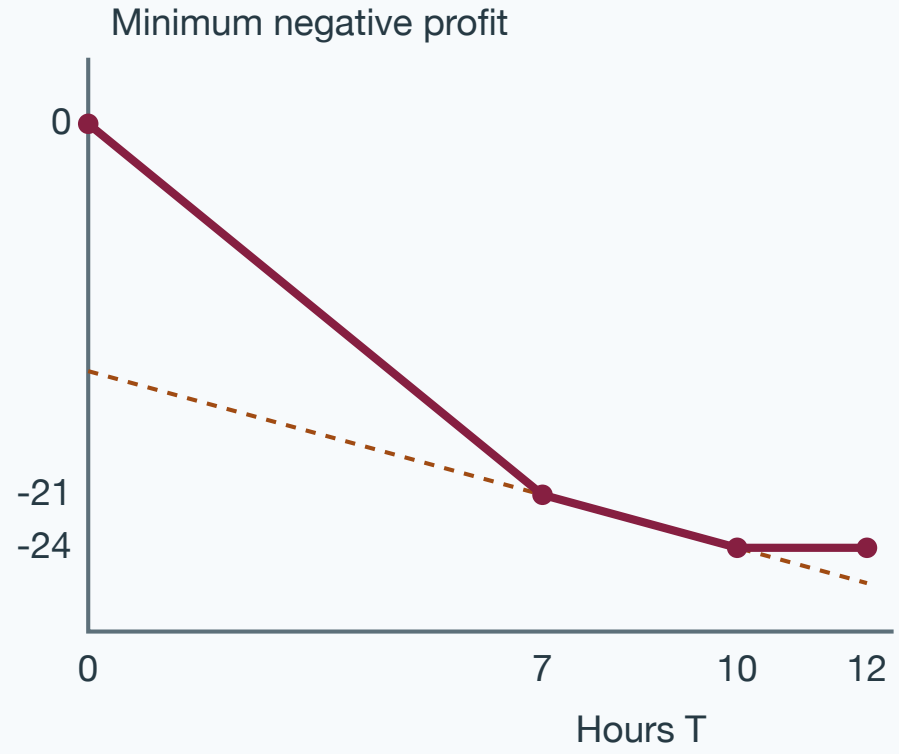
The bakery's maximum profit has the opposite sign, so its graph is concave in resource limits. Do not extend one piece's slope beyond its allowable range.

# The marginal value changes when a new resource becomes scarce

Minimize $-2x - 3y$ with $x + y \le T$, $2x + y \le 16$, $x + 2y \le 14$, $x, y \ge 0$, $T \ge 0$.

$$F(T) = \begin{cases} -3T, & 0 \le T \le 7, \\ -T - 14, & 7 \le T \le 10, \\ -24, & T \ge 10. \end{cases}$$

At $T = 9$, the slope is $-1$: one more hour reduces negative profit by 1, or increases profit by 1.




The dashed line –T–14 is the global lower bound supplied by the dual solution at T=9.

# A dual solution gives a supporting slope even at a corner

For minimization, an optimal dual vector $p^*$ at $b^*$ always gives

$$F(b) \geq F(b^*) + (p^*)^\top (b - b^*).$$

A vector satisfying this inequality is called a **subgradient**: its affine prediction is a global lower bound.

At $T = 7$ in our graph, left slope is $-3$ and right slope is $-1$. Every slope between them supports the graph; there is no single two-sided derivative.

At differentiable points, this supporting vector is the gradient. At corners, do not treat one reported dual value as the unique marginal change in every direction.

Full statement and proof

# Changing costs selects among fixed feasible plans

Keep the original bakery constraints and minimize $-qx - 3y$, where $q$ is bread's profit coefficient.

$$G(q) = \min\{-21,\ -4q - 15,\ -7q - 6,\ -8q\} \quad (q \geq 0).$$

> Each expression is one feasible vertex's cost. The minimum of these affine functions is concave and piecewise linear.

As $q$ crosses $1.5, 3, 6$, the selected production plan changes.




Each line is the cost of a fixed feasible production plan. The minimum selects the best plan.

# Trace the breakpoints instead of solving unrelated cases

For minimization with costs $c + \theta d$, a fixed basis has reduced costs

$$\bar{c}_j(\theta) = c_j + \theta d_j - (c_{\mathcal{B}} + \theta d_{\mathcal{B}})^\top B^{-1} A_j.$$

> Find the interval where all required signs hold. At an endpoint, pivot to a basis valid on the next interval. Ties require the anticycling care already studied.

Cost changes leave primal feasibility intact: use primal simplex. RHS changes leave reduced costs intact: use dual simplex.

Worked breakpoint pivot in the bakery

# Should the bakery pay for extra production time?

Let $x, y$ be bread and cake batches. For this decision, measure profit in **hundreds of dollars**: maximize $z = 2x + 3y$, with $x, y \geq 0$.

**Hours:** $x + y \leq 9$. **Flour:** $2x + y \leq 16$. **Sugar:** $x + 2y \leq 14$.

> Pay **$30 for 30 extra minutes of production time**? Hours change: $9 \to 9.5$, so $\delta = 0.5$.

For $-2 \leq \delta \leq 1$, the original basis stays optimal: $z_{\text{opt}} = 23 + \delta$. Thus $\delta = 0.5$ gives $z_{\text{opt}} = 23.5$, **before the extra fee**.

| Compare the extra profit and fee | Dollars |
|---|---|
| Increase in maximum profit | $(23.5 - 23) \times \$100 = \$50$ |
| Fee for the extra production time | $30 |
| Net improvement | **$50 – $30 = $20: accept.** |

**Why the range matters:** 3 extra hours give only $z_{\text{opt}} = 24$, a $100 increase before any fee—not the $300 predicted by extending the old slope.

# Different production policies can change the scarce resource

In computer production, configurations consume memory boards and disk drives. Demand requirements and permitted substitute boards determine the feasible plans.

| Question | Sensitivity analysis helps answer |
|---|---|
| **Buy more of which component?** | Compare dual values, units, and allowable increases. |
| **Allow a substitute component?** | Add its activity or column and check its reduced cost. |
| **Change a demand policy?** | Rebuild the affected constraints and reoptimize. |

> A resource price belongs to a particular model and basis. A price reported under one policy cannot automatically be used under another.

# Practice: decide which optimality check survives

$x$: bread batches; $y$: cake batches. Start each question from this original problem:

$$\begin{aligned} \max \quad & z = 2x + 3y \\ \text{s.t.} \quad & x + y \le 9 && \text{(hours)}, \\ & 2x + y \le 16 && \text{(flour)}, \\ & x + 2y \le 14 && \text{(sugar)}, \\ & x, y \ge 0. \end{aligned}$$

Unused resources: $s_1$ = hours, $s_2$ = flour, $s_3$ = sugar. The original optimal dictionary is

$$x = 4 - 2s_1 + s_3, \qquad s_2 = 3 + 3s_1 - s_3,$$
$$y = 5 + s_1 - s_3, \qquad z = 23 - s_1 - s_3.$$

| Independent change | Your task · work in pairs for 3 minutes |
|---|---|
| **Hours increase from 9 to 9.5** | Find $x, y$, the unused flour $s_2$, and profit $z$. |
| **Bread profit coefficient increases from 2 to 3.2** | Decide whether this basis remains optimal. |
| **Add the bread limit $x \le 3$** | Choose a reoptimization method and explain why. |

# Practice answers: feasibility and objective signs play different roles

| Change | Result |
|---|---|
| **Hours +0.5** | $(x, y, s_2) = (5, 4.5, 1.5)$; profit $23.5$. |
| **Bread profit 3.2** | $\theta = 1.2$; coefficient of $s_3$ becomes $0.2 > 0$. Use primal simplex. |
| **Add x ≤ 3** | New slack $s_4 = -1$; old reduced costs still have correct signs. Use dual simplex. |

> Always identify the changed data and the surviving property before choosing an algorithm.

# Study guide: a range is a claim about one basis

**Be able to:** update a dictionary after an RHS or cost change; check combined changes; interpret a solver report; choose primal or dual simplex; distinguish local predictions from global value functions.

Read Bertsimas and Tsitsiklis, Chapter 5. The appendix contains complete derivations, a second minimization example, and proofs of the global statements.

- Basis formulas
- 100% rule proof
- Changing A
- Value-function proofs
- Second worked example

# Appendix: calculations and proofs

These pages retain the algebra behind the examples and establish the general statements.

- Derive the bakery dictionary
- Complete reoptimization pivots
- Matrix formulas and simultaneous changes
- Convexity, concavity, and supporting slopes
- A minimization example

# Derive the optimal dictionary by subtraction

$$x + y + s_1 = 9, \qquad x + 2y + s_3 = 14.$$

Subtract the first equality from the second, then substitute back:

$$y = 5 + s_1 - s_3, \qquad x = 9 - s_1 - y = 4 - 2s_1 + s_3.$$

$$s_2 = 16 - 2x - y = 3 + 3s_1 - s_3,$$
$$z = 2x + 3y = 23 - s_1 - s_3.$$

All variables are nonnegative. The objective is at most $23$, attained when $s_1 = s_3 = 0$.

# Derive all RHS changes in one dictionary

Write limits $(9+\delta_1, 16+\delta_2, 14+\delta_3)$. Subtract the first resource equation from the third, then substitute:

$$
\begin{aligned}
x &= 4 + 2\delta_1 - \delta_3 - 2s_1 + s_3, \\
y &= 5 - \delta_1 + \delta_3 + s_1 - s_3, \\
s_2 &= 3 - 3\delta_1 + \delta_2 + \delta_3 + 3s_1 - s_3, \\
z &= 23 + \delta_1 + \delta_3 - s_1 - s_3.
\end{aligned}
$$

> The RHS changes only constants. The objective coefficients of nonbasic variables remain –1, –1.

# Full dictionary after adding a product

Maximize $2x + 3y + 4t$, subject to $x + y + t \leq 9$, $2x + y + 2t \leq 16$, $x + 2y + t \leq 14$, $x, y, t \geq 0$.

$$
\begin{aligned}
x &= 4 - 2s_1 + s_3 - t, \\
y &= 5 + s_1 - s_3, \\
s_2 &= 3 + 3s_1 - s_3, \\
z &= 23 - s_1 - s_3 + 2t.
\end{aligned}
$$

Increase $t$. Only $x$ decreases, so the step is $4$: $t$ enters and $x$ leaves.

# The new product replaces bread in one pivot

$$
\begin{aligned}
t &= 4 - x - 2s_1 + s_3, \\
y &= 5 + s_1 - s_3, \\
s_2 &= 3 + 3s_1 - s_3, \\
z &= 31 - 2x - 5s_1 + s_3.
\end{aligned}
$$

The point $(x, y, t) = (0, 5, 4)$ has profit $31$, but the positive $s_3$ coefficient means it is **not yet optimal**.

Let $s_3$ enter. Both $y$ and $s_2$ decrease; $s_2$ reaches zero first at step $3$.

$$
(x, y, t) = (0, 2, 7), \qquad z = 34.
$$

Final dictionary and certificate

# Finish the new-product optimization

From $s_2 = 3 + 3s_1 - s_3$, solve $s_3 = 3 + 3s_1 - s_2$ and substitute:

$$
\begin{aligned}
t &= 7 - x + s_1 - s_2, \\
y &= 2 - 2s_1 + s_2, \\
s_3 &= 3 + 3s_1 - s_2, \\
z &= 34 - 2x - 2s_1 - s_2.
\end{aligned}
$$

All reduced profits are nonpositive. The optimum is $(0, 2, 7)$, with profit $34$. The dual solution $p = (2, 1, 0)$ gives the same bound $9(2) + 16(1) = 34$.

# Complete repair after adding x ≤ 3

The new row is $s_4 = -1 + 2s_1 - s_3$. Let $s_4$ leave and $s_1$ enter:

$$
\begin{aligned}
s_1 &= \tfrac{1}{2} + \tfrac{1}{2}s_3 + \tfrac{1}{2}s_4, \\
x &= 3 - s_4, \\
y &= \tfrac{11}{2} - \tfrac{1}{2}s_3 + \tfrac{1}{2}s_4, \\
s_2 &= \tfrac{9}{2} + \tfrac{1}{2}s_3 + \tfrac{3}{2}s_4, \\
z &= \tfrac{45}{2} - \tfrac{3}{2}s_3 - \tfrac{1}{2}s_4.
\end{aligned}
$$

Thus $(x, y) = (3, 5.5)$ is optimal with profit $22.5$. The new constraint reduces the best profit.

# Basis formulas: equality constraints first

For $\min c^\top x$ subject to $Ax = b, x \geq 0$, partition columns into invertible $B$ and nonbasic block $N$.

$$Bx_{\mathcal{B}} + Nx_{\mathcal{N}} = b \quad \Rightarrow \quad x_{\mathcal{B}} = B^{-1}b - B^{-1}Nx_{\mathcal{N}}.$$

$$c^\top x = c_{\mathcal{B}}^\top B^{-1}b + \underbrace{(c_{\mathcal{N}}^\top - c_{\mathcal{B}}^\top B^{-1}N)}_{\bar{c}_{\mathcal{N}}^\top} x_{\mathcal{N}}.$$

An optimal basis requires $B^{-1}b \geq 0$ and $\bar{c}_{\mathcal{N}} \geq 0$. For maximization, reverse only the reduced-cost sign.

# A general one-at-a-time RHS interval

Change $b_i$ by $\delta$. Define $g = B^{-1}e_i$, where $e_i$ has 1 in position $i$ and zeros elsewhere.

$$x_{\mathcal{B}}(\delta) = x_{\mathcal{B}} + \delta g.$$

$$\max_{k:g_k>0} \frac{-x_{\mathcal{B},k}}{g_k} \le \delta \le \min_{k:g_k<0} \frac{-x_{\mathcal{B},k}}{g_k}.$$

An empty lower set gives $-\infty$; an empty upper set gives $+\infty$. Rows with $g_k = 0$ impose no new restriction.

The reduced costs do not change. With $p^\top = c_{\mathcal{B}}^\top B^{-1}$, the optimal value inside this interval is $F(b) + \delta p_i$.

# Why the 100% rule guarantees feasibility

Let $d_i$ be the positive finite allowable magnitude in the chosen direction, and $v_i$ the signed change of basic values per unit. Then $x_{\mathcal{B}} + d_i v_i \geq 0$.

Set $\alpha_i = |\delta_i|/d_i$. If $R = \sum_i \alpha_i \leq 1$, then

$$x_{\mathcal{B}} + \sum_i |\delta_i| v_i = \left(1 - \sum_i \alpha_i\right) x_{\mathcal{B}} + \sum_i \alpha_i (x_{\mathcal{B}} + d_i v_i) \geq 0.$$

> This is a nonnegative combination of feasible basic-value vectors. Directions with infinite allowance have nonnegative components and contribute zero to R. A nonzero change with zero allowance is not certified.

The rule is sufficient, not necessary: changes can offset one another even when the percentage sum exceeds 100%.

Worked bakery example above 100%

# General cost changes: include the changing basic costs

For minimization, change $c$ to $c + \Delta c$. The quantities stay $B^{-1}b$, while

$$\bar{c}_{\mathcal{N}}^{\text{new}} = \bar{c}_{\mathcal{N}} + \Delta c_{\mathcal{N}} - N^{\top} B^{-\top} \Delta c_{\mathcal{B}}.$$

> Nonbasic cost $c_j$ changes alone: require $\bar{c}_j + \delta \geq 0$. Basic cost in basis position $k$ changes: require $\bar{c}_j - \delta (B^{-1} A_j)_k \geq 0$ for every nonbasic $j$.

The new value, while this basis remains optimal, is $(c_{\mathcal{B}} + \Delta c_{\mathcal{B}})^{\top} x_{\mathcal{B}}$.

# Changing a nonbasic column: retain its old reduced cost

For minimization, replace nonbasic column $A_j$ by $A_j + \delta e_i$. The basis and basic values are unchanged.

$$\begin{aligned} \bar{c}_j^{\text{new}} &= c_j - p^\top (A_j + \delta e_i) \\ &= \bar{c}_j - \delta p_i. \end{aligned}$$

Require $\bar{c}_j - \delta p_i \geq 0$. If $p_i > 0$, this is $\delta \leq \bar{c}_j / p_i$. If $p_i < 0$, reverse the inequality. If $p_i = 0$, this entry change has no effect on that reduced cost.

All other reduced costs stay unchanged.

# A basic-column change is not generally linear in size

In the bakery, replace the hours constraint by $(1+\varepsilon)x + y \le 9$, keeping the other constraints and profits.

Keeping hours and sugar active gives $(1+\varepsilon)x + y = 9$ and $x + 2y = 14$, hence

$$x = \frac{4}{1+2\varepsilon}, \quad y = 7 - \frac{2}{1+2\varepsilon}, \quad s_2 = 9 - \frac{6}{1+2\varepsilon}.$$

$$z(\varepsilon) = 21 + \frac{2}{1+2\varepsilon}, \qquad z'(0) = -4 = -p_1 x^*.$$

For $\varepsilon \ge -1/6$, this basis is feasible and dual feasible. Below that limit, recheck or reoptimize.

# Why the coefficient-change derivative is $-p_i x_j$

For a basic column in position $k$, let $B(\delta) = B + \delta e_i e_k^\top$. Suppose this basis remains invertible, feasible and dual feasible for small $\delta$. Let $v(\delta) = c_{\mathcal{B}}^\top x_{\mathcal{B}}(\delta)$ be the optimal value.

$$B(\delta)x_{\mathcal{B}}(\delta) = b \quad \Rightarrow \quad e_i x_j^* + B x'_{\mathcal{B}}(0) = 0.$$

$$v'(0) = c_{\mathcal{B}}^\top x'_{\mathcal{B}}(0) = -c_{\mathcal{B}}^\top B^{-1} e_i x_j^* = -p_i x_j^*.$$

Strict basic values and strict nonbasic reduced-cost signs are sufficient to keep the basis optimal for sufficiently small changes. At degeneracy or a breakpoint, a unique derivative is not automatic.

# RHS convexity: combine two feasible optimal plans

Fix $A, c$, and assume a finite optimum at feasible RHS vectors $b^1, b^2$. Let their optimal plans be $x^1, x^2$. For $0 \le \lambda \le 1$, set $\hat{x} = \lambda x^1 + (1-\lambda)x^2$.

$$A\hat{x} = \lambda b^1 + (1-\lambda)b^2, \qquad \hat{x} \ge 0.$$

$$\begin{aligned} F(\lambda b^1 + (1-\lambda)b^2) &\le c^\top \hat{x} \\ &= \lambda F(b^1) + (1-\lambda)F(b^2). \end{aligned}$$

This is exactly the convexity inequality.

# Why there are finitely many linear pieces

Assume $A$ has full row rank and the dual set $D = \{p : A^\top p \le c\}$ is nonempty. It has no lines and has finitely many vertices $p^1, \dots, p^K$.

For each feasible RHS with finite optimum, a dual vertex attains the maximum. Strong duality gives

$$F(b) = \max_{k=1,\dots,K} (p^k)^\top b.$$

> Each vertex supplies a linear function. Their maximum is convex and piecewise linear. Several may agree at a breakpoint.

Dependent equality rows can be removed first. Do not infer a unique dual vector merely because a plotted one-dimensional segment is straight.

# An optimal dual vector is a subgradient

Extend $F(b)$ to $+\infty$ outside $S = \{Ax : x \geq 0\}$, and assume $D \neq \varnothing$. Let $p^* \in D$ attain the dual optimum at $b^*$.

$$\begin{aligned} F(b) &\geq (p^*)^\top b \\ &= F(b^*) + (p^*)^\top (b - b^*). \end{aligned}$$

> This follows directly from weak duality and equality of primal and dual values at the reference RHS.

A **subgradient** is any vector giving this supporting inequality for every $b$. The next page proves the converse.

# Every subgradient is an optimal dual vector

Suppose $q$ satisfies the supporting inequality at $b^*$. Because $F(0) = 0$ and $F(2b^*) = 2F(b^*)$, substitution of these two RHS vectors gives

$$q^\top b^* \geq F(b^*), \qquad q^\top b^* \leq F(b^*).$$

Thus $q^\top b^* = F(b^*)$. For each column $A_j$, the plan $x = e_j$ is feasible and costs $c_j$. Therefore

$$c_j \geq F(A_j) \geq F(b^*) + q^\top (A_j - b^*) = q^\top A_j.$$

So $A^\top q \leq c$ and $q^\top b^* = F(b^*)$: $q$ is dual optimal.

# Cost concavity: minimize over a fixed feasible set

Fix nonempty $P = \{x : Ax = b, x \geq 0\}$. On costs with finite optimum, let $G(c) = \min_{x \in P} c^\top x$.

$$\begin{aligned} G(\lambda c^1 + (1-\lambda)c^2) &= \min_{x \in P} \{\lambda (c^1)^\top x + (1-\lambda)(c^2)^\top x\} \\ &\geq \lambda G(c^1) + (1-\lambda) G(c^2). \end{aligned}$$

A basic feasible optimum exists, so $G(c) = \min_k c^\top x^k$ over the finite set of basic feasible solutions. It is concave and piecewise linear.

> An optimal $x^*$ at $c^*$ gives an upper supporting bound: $G(c) \leq G(c^*) + (x^*)^\top (c - c^*)$. Such a vector is a **supergradient**.

# A cost breakpoint: bread profit rises past 3

For maximizing $qx + 3y$, the original basis has

$$z = 4q + 15 + (3 - 2q)s_1 + (q - 3)s_3.$$

At $q = 3$, the coefficient of $s_3$ is zero. Just above 3, let $s_3$ enter. The ratio test gives $s_2$ leaving at step 3.

$$\begin{aligned} s_3 &= 3 + 3s_1 - s_2, \\ x &= 7 + s_1 - s_2, \qquad y = 2 - 2s_1 + s_2, \\ z &= 7q + 6 + (q - 6)s_1 + (3 - q)s_2. \end{aligned}$$

The new basis is optimal for $3 \le q \le 6$. At $q = 6$, the next breakpoint is reached.

# Verify all three RHS-value pieces

Use hours $T \geq 0$, flour 16, sugar 14, and maximize $2x + 3y$.

| Hours | Optimal plan | Matching dual variables |
|---|---|---|
| $0 \leq T \leq 7$ | $(0, T)$, profit $3T$ | $(3, 0, 0)$ |
| $7 \leq T \leq 10$ | $(2T - 14, 14 - T)$, profit $T + 14$ | $(1, 0, 1)$ |
| $T \geq 10$ | $(6, 4)$, profit $24$ | $(0, 1/3, 4/3)$ |

Every listed plan is feasible on its interval and attains its dual upper bound. Negate profits and dual variables to obtain the minimization graph.

# Second example: minimization in standard form

$$\begin{aligned} \min \quad & 2x_1 + 5x_2 + 8x_3 + 6x_4 \\ \text{s.t.} \quad & x_1 + x_2 + 2x_3 = 5, \\ & x_2 + x_3 + x_4 = 3, \qquad x \geq 0. \end{aligned}$$

$$\begin{aligned} x_1 &= 2 - x_3 + x_4, \\ x_2 &= 3 - x_3 - x_4, \\ f &= 19 + x_3 + 3x_4. \end{aligned}$$

Basic variables $x_1, x_2$; optimum $(2, 3, 0, 0)$, cost $19$. The nonnegative objective coefficients have the minimization sign.

# Second example: change the second RHS

Replace $3$ by $3+\delta$, with $x_3 = x_4 = 0$.

$$x_2 = 3+\delta, \qquad x_1 = 2-\delta.$$

$$-3 \le \delta \le 2, \qquad f^* = 2(2-\delta) + 5(3+\delta) = 19 + 3\delta.$$

Dual variables $p = (2, 3)$ are unchanged. At $\delta = -3$, $x_2 = 0$; at $\delta = 2$, $x_1 = 0$.

For simultaneous changes $\Delta b$, the basic values are $(2 + \Delta b_1 - \Delta b_2,\ 3 + \Delta b_2)$; test both together.

# Second example: nonbasic and basic costs

Original objective dictionary: $f = 19 + x_3 + 3x_4$.

Change nonbasic cost $c_3 = 8 + \theta$: $f = 19 + (1 + \theta)x_3 + 3x_4$. Require $\theta \geq -1$, or $c_3 \geq 7$.

Change basic cost $c_1 = 2 + \theta$: substitute $\theta x_1 = \theta(2 - x_3 + x_4)$.

$$f = 19 + 2\theta + (1 - \theta)x_3 + (3 + \theta)x_4.$$

Require $-3 \leq \theta \leq 1$. Quantities stay $(2, 3, 0, 0)$; the dual vector becomes $(2 + \theta, 3 - \theta)$.

# Second example: verify the matrix and dual formulas

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad B^{-1}b = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$p^{\top} = (2,5)B^{-1} = (2,3).$$

$$\bar{c}_3 = 8 - (2,3)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1, \qquad \bar{c}_4 = 6 - (2,3)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3.$$

The same numbers appear in the objective dictionary. Matrix sensitivity and dictionary substitution express the same operations.

# Degeneracy can make reported ranges basis-dependent

Consider $\max x$ subject to $x \le 1$, $x \le 1$, $x \ge 0$. The optimum is $x = 1$. Both constraints are active.

$$\min \; p_1 + p_2 \quad \text{s.t. } p_1 + p_2 \ge 1, \quad p_1, p_2 \ge 0.$$

Every $(p_1, p_2) = (t, 1 - t)$, $0 \le t \le 1$, is dual optimal. There is no unique price for one duplicated constraint.

> Increase only the first RHS: the second row still limits $x$ to 1. Decrease the first RHS: it immediately becomes the limiting row. The value has a corner.

A reported zero allowable change or an active row with zero dual variable is therefore mathematically meaningful.

# Above 100%: check the dictionary directly

**Bakery:** maximize $z = 2x + 3y$, with $x + y \leq 9$, $2x + y \leq 16$, $x + 2y \leq 14$, $x, y \geq 0$.

Return to the original basis $(x, y, s_2)$. Now increase **hours by 1** and **sugar by 3**; keep flour at 16. Their one-at-a-time allowable increases are 1 and 4:

Sum of fractions: $R = \frac{1}{1} + \frac{3}{4} = \frac{7}{4} > 1$.

The changed limits are $(10, 16, 17)$. Substitution gives the full dictionary:

$$
\begin{aligned}
x &= 3 - 2s_1 + s_3, \\
y &= 7 + s_1 - s_3, \\
s_2 &= 3 + 3s_1 - s_3, \\
z &= 27 - s_1 - s_3.
\end{aligned}
$$

At $s_1 = s_3 = 0$, $(x, y, s_2) = (3, 7, 3) \geq 0$. The objective coefficients remain nonpositive: **the original basis is still optimal**, despite $R > 1$.

Return to the 100% rule