Duality IV: Farkas' Lemma and LP Optimality Conditions

Can the constraints be satisfied? Farkas' lemma gives a short certificate when the answer is no.

Is a feasible point optimal? LP optimality conditions combine feasibility and complementary slackness.

We apply these ideas to fractional knapsack and a finite-state financial model.

Reading: Bertsimas–Tsitsiklis §4.6. Complete proofs and supplementary examples

From a failed dual-simplex row to a contradiction

Suppose a dictionary for nonnegative variables contains

$$x_3 = -1 - 2x_1 - 3x_2.$$

Rearrange that row:

$$\underbrace{2x_1 + 3x_2 + x_3}_{\geq 0} = -1.$$

Every term on the left is nonnegative. No feasible point can satisfy this equation.

Next: Can we obtain an impossible equation directly by combining the original constraints?

Can we prove infeasibility directly from the original constraints?

Consider a new example with nonnegative variables:

$$5x_1 + x_2 + 3x_3 = 1, \qquad 3x_1 + x_2 = 2, \qquad x \geq 0.$$

$$A = \begin{pmatrix} 5 & 1 & 3 \\ 3 & 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

We ask only whether both equalities can hold. There is no objective to optimize.

> Subtract the second equality from the first. What equation must every feasible point satisfy?

A subtraction proves there is no feasible point

$$5x_1 + x_2 + 3x_3 = 1, \qquad 3x_1 + x_2 = 2, \qquad x \geq 0.$$

$$(5x_1 + x_2 + 3x_3) - (3x_1 + x_2) = 1 - 2$$

$$2x_1 + 3x_3 = -1.$$

The left side is nonnegative. The right side is negative. Therefore the constraints are inconsistent with nonnegative variables.

A vector records which equations we combined

$$5x_1 + x_2 + 3x_3 = 1, \qquad 3x_1 + x_2 = 2, \qquad x \geq 0.$$

The vector $q = (1, -1)^\top$ records the multipliers: first equality times 1, second equality times -1, then add.

$$q^\top A = (2, 0, 3) \geq 0, \qquad q^\top b = 1 - 2 = -1.$$

$$0 \leq 2x_1 + 3x_3 = q^\top A x = q^\top b = -1.$$

We call q an **infeasibility certificate**: it records how combining the constraints gives an impossible equation for $x \geq 0$.

The same certificate separates the target from the columns

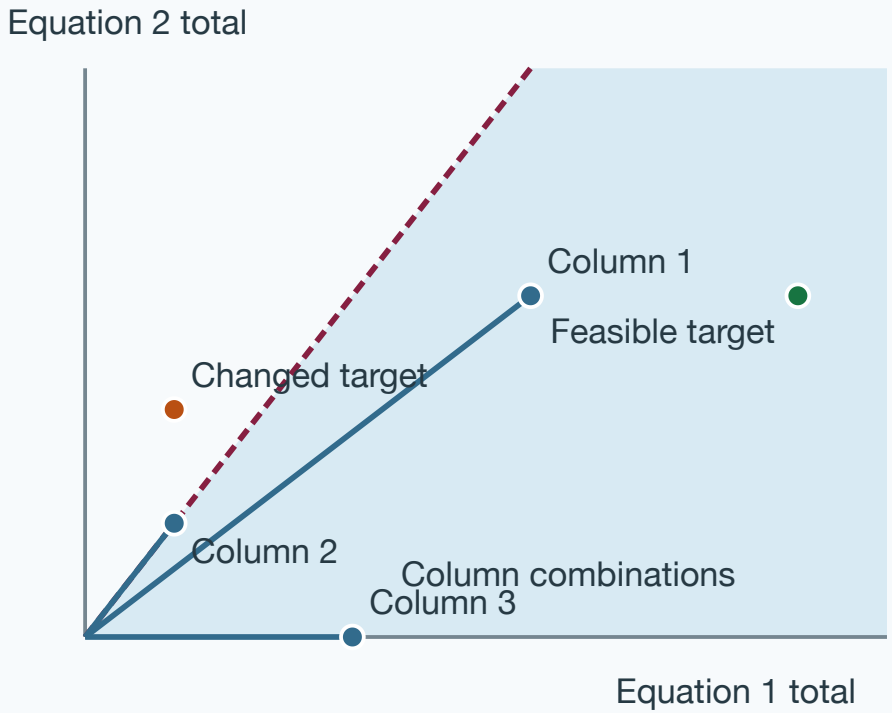
Columns: $A_1 = (5, 3)$, $A_2 = (1, 1)$, $A_3 = (3, 0)$; target $b = (1, 2)$.

A feasible point would express b as a nonnegative combination of columns:

$$b = x_1 A_1 + x_2 A_2 + x_3 A_3.$$

> Each column has first coordinate minus second coordinate nonnegative. So does every nonnegative combination.

The target $(1, 2)$ has difference -1, so it lies outside that cone.



Dashed line: the two totals are equal. The green target $(8, 3)$ is attainable; the orange target $(1, 2)$ is not.

Theorem 4.6: Farkas' lemma

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Feasible system

$$\text{There exists } x \geq 0 \text{ with } Ax = b.$$

Infeasibility certificate

$$\text{There exists } q \in \mathbb{R}^m : \quad A^\top q \geq 0, \quad b^\top q < 0.$$

The entries of q are unrestricted. The two alternatives cannot both hold, and one must hold.

Proof in the appendix

Practice: verify a certificate

Use the same A, but now $b = (2, 3)^\top$:

$$5x_1 + x_2 + 3x_3 = 2, \qquad 3x_1 + x_2 = 3, \quad x \geq 0.$$

> Try $q = (1, -1)^\top$. Compute $A^\top q$ and $b^\top q$. What do they prove?

Write the resulting scalar equation as well as the matrix products.

The certificate is a checkable contradiction

$$A^\top q = (2, 0, 3)^\top \geq 0, \qquad b^\top q = 2 - 3 = -1 < 0.$$

$$2x_1 + 3x_3 = -1 \quad \text{is impossible for } x \geq 0.$$

For the old target $b = (8, 3)^\top$, the same q gives $b^\top q = 5$. It is not an infeasibility certificate there.

Indeed, $x = (1, 0, 1)$ gives $Ax = (8, 3)^\top$: the old system is feasible.

A zero product does not decide feasibility

Both systems have $x_1, x_2 \geq 0$ and $b = (1, 1)^\top$. Try $q = (1, -1)^\top$:

$$b^\top q = 1 - 1 = 0.$$

Constraints	First equality minus second	Conclusion
$x_1 + x_2 = 1,$ $x_2 = 1.$	$x_1 = 0,$ $A^\top q = (1, 0)^\top \geq 0.$	**Feasible:** $x = (0, 1)$ satisfies both equalities and nonnegativity.
$x_1 + 2x_2 = 1,$ $x_2 = 1.$	$x_1 + x_2 = 0,$ $A^\top q = (1, 1)^\top \geq 0.$	**Infeasible:** $x_2 = 1$ forces $x_1 = -1$, violating nonnegativity.

The certificate test needs $b^\top q < 0$. When the product is zero, the resulting equation can still force variables to zero.

Why a certificate rules out every feasible point

Suppose a certificate q satisfies $A^\top q \geq 0$ and $b^\top q < 0$.

$$\underbrace{(A^\top q)^\top x}_{\geq 0 \text{ when } x \geq 0} = q^\top A x = q^\top b < 0.$$

This contradiction excludes every nonnegative solution of the equalities.

The deeper part of Farkas' lemma is existence: whenever the system is infeasible, some such vector can be found. Phase I and duality prove that part.

Read the existence proof

Inequalities can also give an impossible statement

So far: equalities with $x \geq 0$. Now consider inequalities with **no separate sign restriction** on x.

Write both inequalities with the same direction:

$$x \geq 2, \quad x \leq 1 \qquad \Longleftrightarrow \qquad -x \leq -2, \quad x \leq 1.$$

Multiply each inequality by 1, then add. The vector $q = (1, 1)^\top$ records these multipliers.

$$0 = (-x) + x \leq -2 + 1 = -1.$$

The variable cancels, leaving an impossible inequality. Therefore the original system is infeasible.

Why cancel? Without a sign restriction, $2x \leq -1$ alone is possible: $x = -1$ satisfies it. But $0 \leq -1$ is impossible for every x.

Three conditions make the same argument work in general

Consider $Ax \leq b$, with $x \in \mathbb{R}^n$ unrestricted (no separate sign restrictions). Let q_i be the multiplier for inequality i.

Condition on the multipliers	Why we need it
$q \geq 0$	Multiplying and adding the inequalities preserves "≤".
$A^\top q = 0$	Every variable cancels from the combined left-hand side.
$b^\top q < 0$	The combined right-hand side is negative.

$$0 = (A^\top q)^\top x = q^\top A x \leq q^\top b < 0.$$

The alternative: exactly one exists: an x satisfying $Ax \leq b$, or a vector q satisfying all three conditions above.

Why such multipliers always exist for an infeasible system

From “no feasible point” to “no better feasible point”

Farkas certificate: a combination of constraints proves infeasibility.

Optimality certificate: a dual feasible point gives a bound; a primal feasible point attains it.

We already know primal feasibility, dual feasibility, and complementary slackness.

Taken together, these are the LP **Karush–Kuhn–Tucker (KKT) conditions**. We will show why they are both necessary and sufficient for optimality.

LP KKT conditions in standard form

$$\begin{aligned} \text{Primal:} \quad & \min c^\top x \quad \text{s.t. } Ax = b,\ x \geq 0, \\ \text{Dual:} \quad & \max b^\top p \quad \text{s.t. } A^\top p \leq c,\ p \text{ unrestricted.} \end{aligned}$$

Define the **dual slack** $r = c - A^\top p$.

Condition	Equation or inequality
Primal feasibility	$Ax = b, \quad x \geq 0$
Dual feasibility	$A^\top p + r = c, \quad r \geq 0$
Complementary slackness	$x_j r_j = 0 \quad \text{for every } j$

Theorem: a primal feasible x is optimal **if and only if** there exists a dual feasible p with $x_j r_j = 0$ for every j. That p is also optimal.

Why the LP KKT conditions are sufficient and necessary

$$\begin{aligned} \text{Primal:} \quad & \min c^\top x \quad \text{s.t. } Ax = b,\ x \geq 0, \\ \text{Dual:} \quad & \max b^\top p \quad \text{s.t. } A^\top p \leq c,\ p \text{ unrestricted.} \end{aligned}$$

For a feasible pair x, p, let $r = c - A^\top p \geq 0$. Since $Ax = b$,

$$\begin{aligned} c^\top x - b^\top p &= c^\top x - p^\top Ax \\ &= (c - A^\top p)^\top x = \sum_j r_j x_j. \end{aligned}$$

Sufficient — conditions imply optimality. If every $r_j x_j = 0$, the feasible objective values agree. **Weak duality** proves both points are optimal.

Necessary — optimality implies the conditions. If x is optimal, **strong duality** supplies a dual feasible p with the same value. The sum is zero and every term is nonnegative, so every $r_j x_j = 0$.

Inequality constraints add a second set of zero products

$$\begin{aligned} \text{Primal:} \quad & \min c^\top x \quad \text{s.t. } Ax \geq b,\ x \geq 0, \\ \text{Dual:} \quad & \max b^\top p \quad \text{s.t. } A^\top p \leq c,\ p \geq 0. \end{aligned}$$

For a feasible pair, define **primal slack** $s = Ax - b \geq 0$ and **dual slack** $r = c - A^\top p \geq 0$.

$$c^\top x - b^\top p = x^\top r + p^\top s = \sum_j r_j x_j + \sum_i p_i s_i.$$

Both points are optimal if and only if

$$x_j r_j = 0 \text{ for every } j, \qquad p_i s_i = 0 \text{ for every } i.$$

Every term is nonnegative. Zero products imply optimality by weak duality; optimality forces zero products by strong duality.

An inactive primal constraint has zero dual variable; a positive primal variable has an active dual constraint.

Application: Fractional Knapsack

Which wines should the merchant bring? A merchant can bring **8 liters** of wine to an exhibition. Any part of each supply may be brought. Assume everything brought is sold, with profit proportional to volume.

Wine	1	2	3	4
Available volume (liters)	4	3	2	5
Profit from the entire supply ($)	20	12	6	10

Let x_j be the **fraction of the available supply** of wine j brought. For example, $x_3 = \frac{1}{2}$ means **1 liter** and **$3 profit**.

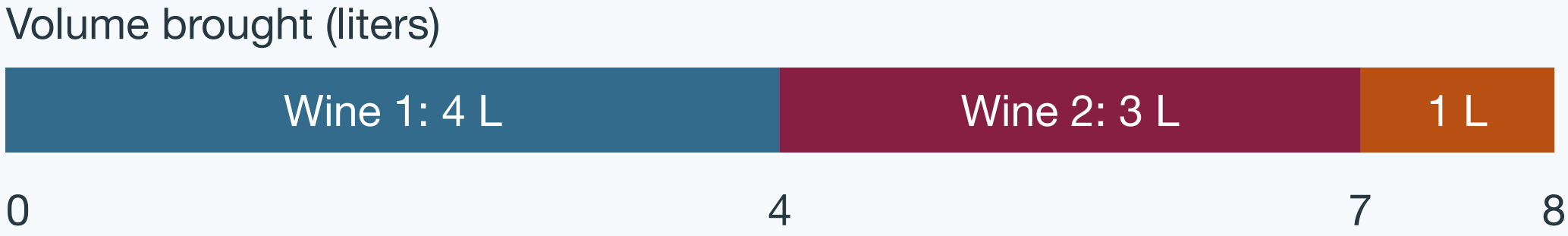
$$\begin{aligned} \max \quad & 20x_1 + 12x_2 + 6x_3 + 10x_4 \\ \text{s.t.} \quad & 4x_1 + 3x_2 + 2x_3 + 5x_4 \leq 8, \\ & 0 \leq x_j \leq 1, \quad j = 1, \dots, 4. \end{aligned}$$

A **fractional knapsack problem** chooses amounts of divisible goods under one capacity limit.

Bring the wines with higher profit per liter first

Maximize $20x_1 + 12x_2 + 6x_3 + 10x_4$, with $4x_1 + 3x_2 + 2x_3 + 5x_4 \leq 8$ and $0 \leq x_j \leq 1$.

Wine	1	2	3	4
Profit per liter ($/L)	$20/4 = 5$	$12/3 = 4$	$6/2 = 3$	$10/5 = 2$



Wine 3: bring 1 of its available 2 liters, so its fraction is one-half.

$$x^* = \left(1, 1, \tfrac{1}{2}, 0\right), \qquad c^\top x^* = 20 + 12 + 3 = 35.$$

Can duality and complementary slackness prove this **$35 allocation is optimal**?

The wine problem and its dual

Maximize $20x_1 + 12x_2 + 6x_3 + 10x_4$, with $4x_1 + 3x_2 + 2x_3 + 5x_4 \le 8$ and $0 \le x_j \le 1$.

The dual variable β corresponds to the **8-liter constraint**; δ_j corresponds to the **supply bound** $x_j \le 1$.

$$
\begin{aligned}
\min \quad & 8\beta + \delta_1 + \delta_2 + \delta_3 + \delta_4 \\
\text{s.t.} \quad & 4\beta + \delta_1 \ge 20, \\
& 3\beta + \delta_2 \ge 12, \\
& 2\beta + \delta_3 \ge 6, \\
& 5\beta + \delta_4 \ge 10, \\
& \beta, \delta_1, \delta_2, \delta_3, \delta_4 \ge 0.
\end{aligned}
$$

Use wine 3 to find the first dual variable

Maximize $20x_1 + 12x_2 + 6x_3 + 10x_4$, with $4x_1 + 3x_2 + 2x_3 + 5x_4 \leq 8$ and $0 \leq x_j \leq 1$.

Proposed allocation: $x = (1, 1, \frac{1}{2}, 0)$. To test optimality, seek a dual solution satisfying complementary slackness with this allocation.

1. Wine 3 is not fully used: $x_3 = \frac{1}{2} < 1$, so its supply bound is inactive.

$$\delta_3(1 - x_3) = 0 \quad \Longrightarrow \quad \delta_3 = 0.$$

2. Wine 3 is used: $x_3 > 0$, so its dual constraint must be an equality.

$$2\beta + \delta_3 = 6 \quad \Longrightarrow \quad 2\beta = 6 \quad \Longrightarrow \quad \boxed{\beta = 3}.$$

Next, find the other dual variables and check that this gives a feasible dual solution.

Complete the dual solution

For $x = (1, 1, \frac{1}{2}, 0)$, we found $\delta_3 = 0$ and $\beta = 3$, the dual variable for the **8-liter constraint**.

Primal information	Complementary slackness	Result
$x_1 = 1 > 0$	$4\beta + \delta_1 = 20$	$\delta_1 = 20 - 4(3) = 8$
$x_2 = 1 > 0$	$3\beta + \delta_2 = 12$	$\delta_2 = 12 - 3(3) = 3$
$x_4 = 0 < 1$	The supply bound $x_4 \leq 1$ is inactive.	$\delta_4 = 0$

Thus $\beta = 3$, $\delta = (8, 3, 0, 0)$, all nonnegative. The first three dual constraints are equalities; the fourth satisfies $5\beta + \delta_4 = 15 \geq 10$.

Dual objective: $8(3) + 8 + 3 = 35$.

Primal profit: $20 + 12 + 6(\frac{1}{2}) = 35$.

Both solutions are feasible and their objective values agree. Weak duality proves both are optimal.

The dual variable predicts profit within a valid range

Keep the wine supplies $a = (4, 3, 2, 5)$ and profits $c = (20, 12, 6, 10)$, but change capacity from 8 liters to b liters. Let $V(b)$ be the maximum profit in dollars.

$$7 \le b \le 9: \quad x = \left(1, 1, \frac{b-7}{2}, 0\right), \quad V(b) = 11 + 3b.$$

For capacity changes within 7–9 liters, each extra liter is filled with wine 3 and adds **$3 profit**. The certificate $\beta = 3, \delta = (8, 3, 0, 0)$ remains valid.

At $b = 10$ liters, bring some wine 4: $x = (1, 1, 1, \frac{1}{5})$, $V(10) = 40$, and $\beta = 2$. Extra capacity no longer earns $3 per liter.

General greedy proof and boundary cases

Application: Arbitrage

An **asset** pays money next year in one of two scenarios (called **states**). Its final payment is its **payoff**. Consider these **proposed prices**:

	Asset 1	Asset 2
Quoted price today	$4	$5
Final payment: state 1	$8	$6
Final payment: state 2	$7	$3

These are final payments, not future trading prices.

> **Buy:** pay today's price and receive the final payment.
>
> **Sell short:** borrow an asset from a lender and sell it to a buyer. In this model, you owe the lender the asset's final payment next year.

Trading assumption: a buyer and a seller are available at each quoted price. We may buy or sell short any amount, with no transaction fees.

The trade exposes inconsistent prices

Buy 1 unit of asset 1 and sell 1 unit of asset 2 short. Money received (+) or paid (–), in dollars:

Trade	Today	State 1 next year	State 2 next year
Buy 1 unit of asset 1	–$4	+$8	+$7
Sell 1 unit of asset 2 short	+$5	–$6	–$3
Combined	+$1	+$2	+$4

We receive **$1 today**. In state 1, we receive $8 and pay the lender $6, leaving **$2**. In state 2, we receive $7 and pay the lender $3, leaving **$4**.

This is an **arbitrage**: no net initial spending, no negative future net payment, and a positive gain. Asset 1 costs less and pays more than asset 2 in both states.

The quotes fail the no-arbitrage test. If no buyer will pay $5 for asset 2, the short sale cannot be executed at $5. The example assumes the trades are available; it tests whether their prices allow arbitrage.

Write the same trade in matrix notation

A **portfolio** h lists the signed number of units of each asset: positive means buy; negative means sell short. Vector c lists today's prices.

In R, **rows are states and columns are assets**. For example, $R_{21} = 7$ is asset 1's payment in state 2.

$$c = \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \qquad R = \begin{pmatrix} 8 & 6 \\ 7 & 3 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$$\underbrace{c^\top h}_{\text{initial cost}} = 4 - 5 = -1, \qquad \underbrace{Rh}_{\text{future payments}} = \begin{pmatrix} 8(1) + 6(-1) \\ 7(1) + 3(-1) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Cost **–$1**: **receive $1** today. Net payments: **$2 in state 1** or **$4 in state 2**.

> In general, an **arbitrage** satisfies $c^\top h \leq 0$ and $Rh \geq 0$, with at least one strict inequality: negative cost, or a positive payoff in at least one state.

Search for money today with no future loss

Minimize today's cost while the net payments in state 1 and state 2 stay nonnegative:

$$\begin{aligned} \min \quad & 4h_1 + 5h_2 \\ \text{s.t.} \quad & 8h_1 + 6h_2 \geq 0, \\ & 7h_1 + 3h_2 \geq 0. \end{aligned}$$

In matrix notation: $\min_h \{c^\top h : Rh \geq 0\}$, with h unrestricted. A **negative cost** means money received today.

> Our trade $h = (1, -1)$ costs -1 and pays $(2, 4)$. Scaling to $h = t(1, -1)$, $t \geq 0$, gives cost $-t$ and payments $(2t, 4t)$: the LP is **unbounded below**.

This LP searches for money received today. An optimum of 0 alone does not exclude a zero-cost trade with a positive future payment.

Write the dual: one variable for each state

Same primal: minimize $4h_1 + 5h_2$, with $8h_1 + 6h_2 \geq 0$, $7h_1 + 3h_2 \geq 0$, and h_1, h_2 unrestricted.

Let p_1 and p_2 be the **dual variables** for the state-1 and state-2 constraints, respectively.

$$\begin{aligned} \max \quad & 0p_1 + 0p_2 \\ \text{s.t.} \quad & 8p_1 + 7p_2 = 4, \\ & 6p_1 + 3p_2 = 5, \\ & p_1, p_2 \geq 0. \end{aligned}$$

Zero right-hand sides give objective 0. The primal's $\geq$ constraints give $p \geq 0$; unrestricted h_1, h_2 give dual equalities.

> Subtract the second equality from the first: $2p_1 + 4p_2 = 4 - 5 = -1$. This is impossible for $p \geq 0$, so the **dual is infeasible**.

Change only the prices: now a certificate exists

Change today's prices from $(4, 5)$ to $c = (7.5, 4.5)^\top$. Keep the payments $R = \begin{pmatrix} 8 & 6 \\ 7 & 3 \end{pmatrix}$.

Primal: $\min_h \{7.5h_1 + 4.5h_2 : Rh \geq 0\}$, with h unrestricted. **Dual:** maximize 0 with $p \geq 0$ and

$$\begin{aligned} 8p_1 + 7p_2 &= 7.5, \\ 6p_1 + 3p_2 &= 4.5. \end{aligned}$$

Choose $p_1 = p_2 = \frac{1}{2} > 0$: both equalities hold, so $R^\top p = c$. For **every portfolio** h,

$$c^\top h = p^\top (Rh) = \tfrac{1}{2}(Rh)_1 + \tfrac{1}{2}(Rh)_2.$$

> Nonnegative future payments imply nonnegative cost today. If either payment is positive, the cost is strictly positive. **These new prices admit no arbitrage.**

Doing nothing attains cost 0. The primal and dual optimal values are both 0.

What does a state price mean?

Keep $R = \begin{pmatrix} 8 & 6 \\ 7 & 3 \end{pmatrix}$ and today's prices $c = (7.5, 4.5)^\top$. We found the dual solution $p = (0.5, 0.5)^\top$.

A **state price** p_i is today's consistent price of a contract paying **\$1 in state** i and **\$0 in every other state**.

Hypothetical contract	State 1 payment	State 2 payment	Price today
Contract 1	\$1	\$0	p_1 = \$0.50
Contract 2	\$0	\$1	p_2 = \$0.50

Asset 1's payments $(8, 7)$ equal those of 8 copies of contract 1 and 7 copies of contract 2. The same comparison for asset 2 gives:

$$8p_1 + 7p_2 = 7.5, \qquad 6p_1 + 3p_2 = 4.5.$$

These hypothetical contracts explain the pricing equations; we need not assume they are separately traded. State prices are not probabilities and need not sum to 1.

What Farkas guarantees

For a finite payoff matrix R, asset prices c, and unrestricted portfolios h, Farkas' lemma says **exactly one** exists:

$$\begin{aligned} \text{Trade:} \quad & Rh \geq 0, \quad c^\top h < 0, \\ \text{Certificate:} \quad & R^\top p = c, \quad p \geq 0. \end{aligned}$$

The trade receives money today with no negative future payment. The certificate rules out every such trade.

Complete no-arbitrage theorem: also exclude zero-cost portfolios with a positive payment in at least one state.

$$\text{No arbitrage} \iff \exists p > 0 : \; R^\top p = c.$$

Here $p > 0$ means **every component is strictly positive**.

Complete proof · Why zero state prices need care

Three uses of the same duality logic

Question	What to exhibit
Is the equality system infeasible?	$A^\top q \geq 0,\ b^\top q < 0$
Is a feasible point optimal?	A feasible dual point with the same value, or the KKT conditions.
Are the asset prices arbitrage-free?	Strictly positive state prices reproducing every asset price.

> Always state the model, the variable signs, and the precise conclusion before interpreting a certificate.

Next: sensitivity analysis changes the data while tracking feasible bases and valid optimality certificates.

Appendix: proofs and further calculations

- Farkas' lemma from Phase I and duality
- The inequality alternative and finding a certificate
- Fractional-knapsack proof
- Seven-item fractional-knapsack example
- No-arbitrage theorem and the role of positivity
- Risk-neutral probabilities, replication, and completeness

Return to the LP KKT proof in the main lecture

Farkas proof 1: normalize the right-hand side

First assume $b \geq 0$. If a row has $b_i < 0$, multiply that entire equality by -1.

> Let D be diagonal with entries ± 1, chosen so $Db \geq 0$. The transformed system is $DAx = Db$.

A certificate $\tilde{q}$ for the transformed system gives a certificate for the original system by setting $q = D\tilde{q}$:

$$A^\top q = (DA)^\top \tilde{q} \geq 0, \qquad b^\top q = (Db)^\top \tilde{q} < 0.$$

It is therefore enough to prove existence for a nonnegative right-hand side.

Farkas proof 2: solve the auxiliary problem

With $b \geq 0$, Phase I is

$$\min \ \mathbf{1}^\top y \quad \text{s.t. } Ax + y = b, \quad x, y \geq 0.$$

> It is feasible at $x = 0, y = b$, and its objective is bounded below by zero. Therefore it has an optimal solution.

If the optimum is zero, then $y = 0$ and $Ax = b, x \geq 0$ is feasible.

If the original system is infeasible, the Phase I optimum is strictly positive.

Farkas proof 3: the Phase I dual gives the certificate

The dual of $\min\{\mathbf{1}^\top y : Ax + y = b,\ x, y \geq 0\}$ is

$$\max\ b^\top u \quad \text{s.t. } A^\top u \leq 0, \quad u \leq \mathbf{1}, \quad u \text{ unrestricted.}$$

When the original system is infeasible, strong duality gives an optimal u with $b^\top u > 0$.

$$q = -u \quad \Longrightarrow \quad A^\top q \geq 0, \qquad b^\top q < 0.$$

This proves a certificate exists. The contradiction identity in the main slides proves the two alternatives are mutually exclusive.

Derive the inequality alternative

For unrestricted x, write $x = x^+ - x^-$, with $x^+, x^- \geq 0$, and introduce $s \geq 0$:

$$Ax \leq b \quad \Longleftrightarrow \quad Ax^+ - Ax^- + s = b.$$

The equality-form certificate for the columns $[A, -A, I]$ requires

$$A^\top q \geq 0, \quad -A^\top q \geq 0, \quad q \geq 0, \quad b^\top q < 0.$$

$$A^\top q = 0, \quad q \geq 0, \quad b^\top q < 0.$$

This also proves existence, rather than merely showing how to check a proposed vector.

Finding a certificate is itself an LP feasibility problem

Any certificate with $b^\top q < 0$ can be rescaled by the positive number $-1/(b^\top q)$.

$$A^\top q \geq 0, \qquad b^\top q = -1, \qquad q \text{ unrestricted.}$$

So we can search for a certificate using linear equalities and inequalities. A successful solution proves infeasibility; a Phase I solve is another way to obtain one.

The normalization -1 is convenient, not an extra assumption on the original system.

Fractional knapsack: define the threshold item

Consider $\max c^\top x$, $\sum_j a_j x_j \le b$, $0 \le x_j \le 1$, with $a_j > 0, c_j \ge 0$. Sort so $c_1/a_1 \ge \cdots \ge c_n/a_n$.

Assume $0 < b < \sum_j a_j$. Choose k so $\sum_{j<k} a_j \le b \le \sum_{j \le k} a_j$.

$$x_j^* = \begin{cases} 1 & j < k, \\ (b - \sum_{i<k} a_i)/a_k & j = k, \\ 0 & j > k. \end{cases}$$

This is feasible and fills the capacity. At an exact boundary, the selected threshold fraction may equal 0 or 1.

Fractional knapsack: construct a matching dual solution

Set $\beta = c_k/a_k$ and $\delta_j = \max\{c_j - \beta a_j, 0\}$.

Then $\beta, \delta \geq 0$ and $\beta a_j + \delta_j \geq c_j$: dual feasibility holds.

$$b\beta + \sum_j \delta_j = b\beta + \sum_{j<k} (c_j - \beta a_j)$$

$$= \sum_{j<k} c_j + c_k \frac{b - \sum_{j<k} a_j}{a_k} = c^\top x^*.$$

The matching values prove the greedy allocation is optimal.

Tied ratios may allow several optimal allocations; they do not invalidate this certificate.

Fractional knapsack: ties and boundary capacities

Case	A valid optimal choice
$b = 0$	$x = 0$; choose $\beta = \max_j c_j/a_j$, $\delta = 0$.
$b \geq \sum_j a_j$	$x = \mathbf{1}$; choose $\beta = 0$, $\delta = c$.
Equal value / resource	Exchange resource among tied items within their bounds without changing the objective.

If $b < 0$, the model with $a > 0, x \geq 0$ is infeasible. The sorting proof above assumed $c \geq 0$; negative-value items can be omitted when capacity is only an upper bound.

Seven items: a larger fractional-knapsack example

A bag has capacity 100. Items can be divided.

Item	1	2	3	4	5	6	7
Size	25	20	15	10	30	10	20
Value	150	100	80	70	140	110	90
Value / size	6	5	$16/3$	7	$14/3$	11	$9/2$

Order by value per size: 6, 4, 1, 3, 2, 5, 7. The first five use 80 units; take 20/30 of item 5.

$$x^* = (1, 1, 1, 1, \tfrac{2}{3}, 1, 0), \qquad c^\top x^* = \tfrac{1810}{3}.$$

Seven items: verify the dual bound

For sizes $a = (25, 20, 15, 10, 30, 10, 20)$, values $c = (150, 100, 80, 70, 140, 110, 90)$, and capacity 100, choose

$$\beta = \tfrac{14}{3}, \qquad \delta = (\tfrac{100}{3}, \tfrac{20}{3}, 10, \tfrac{70}{3}, 0, \tfrac{190}{3}, 0).$$

Every component of $\beta a + \delta - c$ is zero except the seventh, which equals $10/3$. Thus the dual is feasible.

$$100\beta + \sum_j \delta_j = \tfrac{1400}{3} + \tfrac{410}{3} = \tfrac{1810}{3}.$$

The bound equals the value of the allocation on the preceding page.

Why nonnegative state prices are not always enough

$$R = I_2, \qquad c = (1,0)^\top, \qquad p = (1,0)^\top.$$

We have $R^\top p = c$ and $p \geq 0$, so no portfolio can have both $Rh \geq 0$ and $c^\top h < 0$.

> But $h = (0,1)^\top$ costs zero and pays $(0,1)^\top$. It is an arbitrage under the broader definition.

A zero state price can ignore a positive payoff in that state. Strict positivity excludes that possibility.

No-arbitrage theorem: encode every possible gain

For a finite number of states and unrestricted portfolios, define

$$M = \begin{pmatrix} R \\ -c^\top \end{pmatrix}, \qquad v = Mh.$$

The vector v lists all future payoffs followed by the initial receipt. Arbitrage means v is nonnegative and nonzero.

> Rescaling gives an equivalent system:
>
> $$Mh - v = 0, \qquad \mathbf{1}^\top v = 1, \qquad v \geq 0, \quad h \text{ unrestricted}.$$

The market has no arbitrage exactly when this system is infeasible.

No-arbitrage theorem: apply Farkas to the gain system

Apply Farkas to $Mh - v = 0,\ \mathbf{1}^\top v = 1$, replacing unrestricted h by $h^+ - h^-$.

A certificate consists of u for the first equations and a scalar t for the last one:

$$M^\top u = 0, \qquad -u + t\mathbf{1} \geq 0, \qquad t < 0.$$

Thus $u \leq t\mathbf{1} < 0$. Put $\lambda = -u > 0$; then $M^\top \lambda = 0$.

$$R^\top \lambda_{1:m} = c\lambda_{m+1} \quad \Longrightarrow \quad p = \lambda_{1:m}/\lambda_{m+1} > 0, \quad R^\top p = c.$$

Combined with the main-slide identity, this proves: no arbitrage if and only if strictly positive state prices exist.

Fractional knapsack: why an exchange improves value

Suppose item i has strictly greater value per resource than item j, but $x_i < 1$ and $x_j > 0$.

Transfer $\varepsilon = \min\{a_i(1 - x_i), a_j x_j\} > 0$ resource units from item j to item i.

$$x_i' = x_i + \varepsilon/a_i, \qquad x_j' = x_j - \varepsilon/a_j.$$

$$c^\top x' - c^\top x = \varepsilon \left(\frac{c_i}{a_i} - \frac{c_j}{a_j} \right) > 0.$$

Bounds remain satisfied and resource use is unchanged. Therefore an optimum cannot use a lower-ratio item while leaving room in a strictly higher-ratio item. Tied ratios allow value-preserving exchanges.

Fractional knapsack with arbitrary upper bounds

For $\max c^\top x$, $a^\top x \le b$, $0 \le x_j \le u_j$, assume $a_j > 0$, $u_j > 0$.

> Set $t_j = x_j / u_j$. Then $0 \le t_j \le 1$, with new sizes $a_j u_j$ and values $c_j u_j$. Their ratios remain c_j / a_j.

$$\min \; b\beta + \sum_j u_j \delta_j \quad \text{s.t. } \beta a_j + \delta_j \ge c_j, \quad \beta, \delta \ge 0.$$

The upper-bound complementary product is $\delta_j (u_j - x_j) = 0$. Items with $u_j = 0$ are fixed at zero and can be removed.

Normalize state prices to obtain a probability vector

For strictly positive state prices p, put $C = \sum_i p_i$ and $\pi_i = p_i / C$. Then $\sum_i \pi_i = 1$ and

$$c_j = \sum_i p_i R_{ij} = C \sum_i \pi_i R_{ij}.$$

Separate example: use these payments and prices:

$$R = \begin{pmatrix} 70 & 50 \\ 30 & 50 \end{pmatrix}, \quad c = \begin{pmatrix} 50 \\ 40 \end{pmatrix}.$$

Solving $R^\top p = c$ gives $p = (13/20, 3/20)^\top$.

Hence $C = \frac{4}{5}$ and $\pi = (\frac{13}{16}, \frac{3}{16})$.

C is the price of a \$1 payment in every state if that payment is traded or replicated. These **risk-neutral probabilities** need not be the actual state probabilities.

Price a new payoff by replicating it

Keep $R = \begin{pmatrix} 70 & 50 \\ 30 & 50 \end{pmatrix}$, $c = (50, 40)$. A new asset pays $d = (20, 0)$. Solve $Rh = d$.

$$70h_1 + 50h_2 = 20, \quad 30h_1 + 50h_2 = 0 \quad \Longrightarrow \quad h = (\tfrac{1}{2}, -\tfrac{3}{10}).$$

$$c^\top h = 25 - 12 = 13 = p^\top d.$$

If the new asset cost more than 13, sell it and buy this portfolio. If it cost less, buy it and sell the portfolio. Future payoffs cancel and the initial receipt is positive. Its arbitrage-free price must be 13.

Market completeness means every payoff can be replicated

With m future states, the market is **complete** when every $d \in \mathbb{R}^m$ can be written as Rh. This is equivalent to $\operatorname{rank}(R) = m$.

Under no arbitrage, state prices satisfy $R^\top p = c$, $p > 0$. Full row rank of R makes this solution unique.

If rank is smaller than m, choose nonzero v with $R^\top v = 0$. For sufficiently small $\varepsilon > 0$, both $p + \varepsilon v$ and $p - \varepsilon v$ are positive and reproduce the same asset prices.

Thus, under no arbitrage, completeness is equivalent to uniqueness of state prices. Incomplete markets admit several such vectors.

An incomplete market does not determine every new price

Suppose the only asset pays $(1, 1)$ and costs 1. State prices satisfy $p_1 + p_2 = 1, p_1, p_2 > 0$.

A new asset pays $d = (1, 0)$. Its price $k = p^\top d = p_1$ may be any value strictly between 0 and 1.

> At price 0, the new asset gives a free positive payoff. At price 1, buy the original asset and sell the new asset: zero cost, payoff (0,1). Both endpoints permit arbitrage.

The existing asset cannot replicate (1,0), so the old prices alone do not select one interior price.

A payoff bound is another primal–dual LP pair

To cover a target payoff d in every state, minimize the cost of an unrestricted portfolio h:

$$\min_{h} \; c^{\top} h \quad \text{s.t. } Rh \geq d.$$

Its dual is

$$\max_{p} \; d^{\top} p \quad \text{s.t. } R^{\top} p = c, \quad p \geq 0.$$

The equality dual constraints come from unrestricted holdings. If the covering problem is feasible and has a finite optimum, strong duality gives a matching state-price bound. A replicable payoff has the same price under every feasible state-price vector.